İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

GRAVITY ON BRANE WORLDS

M.Sc. Thesis by
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Department : Physics Engineering

Programme: Physics Engineering

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At last, but not least, I would like to dedicate this work to my partner in life, Suzan KÖSEOĞLU.

May 2004 Ahmet Emir GÜMRÜKÇÜOĞLU

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ABBREVIATIONS

KK	: Kaluza-Klein
RS	: Randall-Sundrum
GR	: General Relativity
ADD	: Arkani-Hamed, Dimoploulos, Dvali
SM	: Standard Model
EW	: Electroweak
Pl	: Planck

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NOTATION AND CONVENTIONS

We use the sign conventions of [40], with a metric with signature (-1,1,1,1). Unless noted, all the quantities are assumed five dimensional.

Here is the list of frequently occuring symbols.

$\mathbf{M_5}$	: Five dimensional manifold
$\mathbf{\Sigma_z}$	: Four dimensional hypersurface
$\mathbf{x^A}$	: Coordinates on M_5
$\mathbf{y^i}$	: Coordinates intrinsic to Σ_z
$\mathbf{z}$	: Extra dimension
$\mathbf{g_{AB}}$	: Bulk metric
$\mathbf{h_{ij}}$	: Induced metric
$\mathbf{L}$	: Compactification radius
$\mathbf{M_{pl}}$	: Planck scale
$\mathbf{M_{EW}}$	: Electroweak scale
dS	: de Sitter
AdS	: Anti-de Sitter
$\mathbf{\Lambda}$	: Cosmological constant
$\mathbf{K_{ij}}$	: Extrinsic curvature
$\mathbf{N^i}$	: Shift vector
$\mathbf{N}$	: Lapse function
$\mathbf{n^A}$	: Normal vector
$\mathbf{\Gamma^A{}_{BC}}$	: Connections of g_{AB}
$\mathbf{\lambda^i{}_{jk}}$	: Connections of h_{ij}
$\mathbf{\nabla_A}$	: Covariant derivative compatible with g_{AB}
$\mathbf{D_i}$	: Covariant derivative compatible with h_{ij}
$\mathfrak{L}_{\vec{N}}$	: Lie Derivative with respect to the shift vector

ZAR DÜNYA ÜZERİNDE GRAVİTASYON

ÖZET

Pek çok modern teoride boyut sayısı alışılagelen 3+1 boyuttan fazladır. Yüksek boyutlardan efektif 3+1 boyutlu teoriyi elde etme mekanizmaların biri olan "zar dünya" senaryoları dünyamızı yüksek boyutlu bir "üst-dünya"nın içindeki bir hiperyüzey olarak betimler. Standart Model etkileşimlerinin zar üzerinde lokalize olması, ancak gravitasyonun yüksek boyutlara da çıkabilmesi sayesinde gravitasyonel etkileşimlerin diğer etkileşimlere göre neden daha zayıf olduğuna dair getirdiği açıklama, 30 TeV mertebesinde enerjilerde yüksek boyut etkilerinin gözlenebilmesi umutlarını doğurmuştur. Bu etkilerin gözlenmesi, zar dünyaları öngören sicim teorilerine de deneysel bir destek sağlayacaktır.

Bu çalışmada, ektra boyutlara evrime izin veren genel bir koordinat sistemi için 4+1 boyutlu bir manifoldu, 3+1 boyutlu zamansal hiperyüzeylere dilimleyerek yüksek boyutlu eğrilik terimlerinin yüzey terimleri cinsinden genel ifadelerini elde ettik. Daha sonra Israel'in sınır koşullarından faydalanılarak 3+1 ve 2+1 boyutlarda gravitasyonel alan denklemlerine ulaştık. Bu denklemler, ektra boyutta ivmenin kaldırılmasıyla Gaussian normal koordinatlarda daha önce bulunmuş sonuçları vermektedir.

Ayrıca kısıt denklemlerini Randall Sundrum zarı için çözerek, zar üzerinde bir dönen kara delik çözümü elde ettik.

GRAVITY ON BRANE WORLDS

SUMMARY

In most of the modern theories, the number of dimensions is larger than the usual 3+1 dimensions. "Brane world" scenarios, which are one of the mechanisms of dimensional reduction to an effective 3+1 theory, describe our world as a hypersurface in a higher dimensional "upper-world". With the help of the localization of Standard Model interactions on the brane and the free gravitational fields accessing the bulk, an explanation is found to the weakness of the gravitational fields compared to others. This raised hope on observing higher dimensional effects at order of 30 TeV. The realization of these observations will serve as an experimental support to the string teories which predict the brane worlds.

In this work, for a general coordinate system which allows acceleration in the extra dimensions, we sliced 4+1 manifold into 3+1 hypersurfaces and obtained higher dimensional curvature quantities in terms of the surface quantities. Then applying Israel's junction conditions, we reached gravitational field equations in 3+1 and 2+1 dimensions. These equations reduce to known ones when acceleration in extra dimensions vanishes.

We then solved the constraint equations for a Randall-Sundrum brane, obtaining a rotating black hole solution on the brane.

1. INTRODUCTION

1.1 A Journey to the Extra Dimensions

Throughout history, the main problem of science was to explain, understand and explore the nature with as little words as possible or, one may say, “to draw the boundaries to cover as little area as possible”. The harder the human mind tried to simplify and restrain nature’s behavior, it always found a way out and confused the biggest minds. In other words, with every abnormal behavior, the mathematical tools became more complex and at each step some unexplored behaviors of nature continued to reveal. Although some behaviors can be summarized by simple mathematical formulae, the physical insights leading to that formula are mostly unexpressable by words. The study of extra dimensions takes a step further by being unimaginable. Even though one could for example, *see* an eleven dimensional object, one would only observe the three dimensional projection of it. That is why the spatial dimensions other than three are referred to as *extra*. But by the beginning of the twentieth century, Physics decided to use the calculus of manifolds as a tool, and everything became more and more complex, yet at the same time, more simple.

In 1909, inspired by Maxwell’s *Electrodynamics* and Einstein’s *Special Relativity*, Minkowski suggested that time and three space were of the same nature and could be united as a four dimensional manifold where physics could be expressed in a simple form [4]. The idea was taken further by Einstein in 1915 [5] with *General Relativity*, where he generalized Minkowski’s idea to all four dimensional manifolds.

After the breakthrough created by the revolutionary ideas of Minkowski and Einstein, other physicists started playing with the number of dimensions of our universe, hoping to solve long standing problems or to make old theories mathematically prettier. The mechanism by which one finds an effective four dimensional theory was more or less the same. These were generally called *Kaluza-*

Klein theories, honoring the first physicists who tried this mechanism. The biggest flaw of this mechanism was the near-impossibility to observe any effect resulting from the high dimensionality of space. Then in the nineties, a new approach became popular strengthened by its compatibility with string theories: the *brane world scenarios*. Now there is hope, a hope to explore the existence of other dimensions, a hope to be sure if the route Physics took was the right one. Human mind still moves on to a sea of complexity without a compass, yet nature gets revealed day by day.

1.2 The Kaluza-Klein Picture[1]

The idea of an extra spatial dimension was fist introduced by Nordström in 1914 [6] and independently by Kaluza in 1921 [7], in an attempt to unify general relativity with electrodynamics in a theory of five dimensions. To be able to recover the four dimensional physics effectively, Kaluza conceived the size of the fifth dimension as really small. As for the reason why there is no observational evidence pointing towards a fifth dimension, both Nordström and Kaluza avoided this question and simply demanded that the metric is independent of the fifth coordinate. This assumption is called the *cylinder condition* which is one of the many cases where physics progressed without knowing why. In 1926, Klein contributed Kaluza's work by giving a physical basis for this condition [8,9]. He showed that the cylinder condition would arise naturally if the fifth coordinate had a circular topology, thus the fields would depend on it periodically and could be Fourier-expanded. Also, he showed that the scale should be small enough, so that the energies above the ground state would be so high that they would be unobservable.

Kaluza's technique and Klein's contribution lead to a new way of exploring the nature. This technique is called *Kaluza-Klein (KK) compactification* and can be applied to any theory with any number of extra dimensions. The manifold formed by the extra dimensions is taken compact, essentially homogeneous and very small. The compactness ensures that the spacetime is effectively four dimensional.

A simple case for Kaluza-Klein Compactification is a (4+1) dimensional manifold where the usual four dimensions of spacetime and the extra spatial dimension are

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[1] The review part is based on [1] and [2]

represented respectively by coordinates x^i (i = 0,1,2,3) and z . To be able to obtain a four dimensional effective theory at low energies, one must limit the coordinate z depending on a parameter called *compactification radius* as $z \in [0, 2\pi L]$. Now, the spacetime dimensions are infinite as usual, but the extra dimension is S^1. For a free massless particle, one can write down the Klein-Gordon equation

$$^{(5)}\Box\phi = 0 \tag{1.1}$$

where $^{(5)}\Box$ represents five dimensional d'Alembertian. Fourier-expanding ϕ in periodic coordinates z shows that the solution of (1.1) is the superposition of these functions

$$\phi_{\vec{p},n} = e^{ip_k x^k} e^{inz/L} \tag{1.2}$$

Using this in (1.1) gives

$$p^i p_i - \frac{n^2}{L^2} = 0 \tag{1.3}$$

From the four dimensional point of view, the second term of this equation is the mass squared. So the mass of a KK mode is of order L^{-1}, and by setting the compactification radius small enough, one can truncate to the massless mode in low energy limit.

Kaluza's aim was to explore five dimensional gravity without any matter fields. Taking the Einstein-Hilbert action in five dimensions

$$S = -\frac{1}{16\pi G_5}\int d^4x \int_0^{2\pi L} dz\sqrt{-g}\ {}^{(5)}R \tag{1.4}$$

where g_{AB} and $^{(5)}R$ are the five dimensional metric and Ricci tensor, respectively. In order to reach the effective four dimensional action, one must first choose the form of the five dimensional metric. Generally, the (i,j) part of g_{AB} is identified with

h_{ij} (four dimensional metric), the (i,4) part with the four dimensional electromagnetic potential tensor A_i and the (4,4) part with a scalar field ϕ. A convenient metric ansatz would be [1]

$$g_{AB} = \begin{pmatrix} h_{ij} + \kappa^2 \phi^2 A_i A_j & \kappa\phi^2 A_i \\ \kappa\phi^2 A_j & \phi^2 \end{pmatrix} \tag{1.5}$$

where κ is a scaling constant used for later convenience. When the cylinder condition is applied, the integral with respect to the fifth coordinate in (1.4) becomes trivial. Then expressing the five dimensional Ricci scalar in terms of the four dimensional one and setting $\kappa = 4\sqrt{\pi G_4}$ one finds [1]

$$S = -\int d^4x\sqrt{-h}\phi\left(\frac{RL}{8G_5} + \frac{1}{4}\phi^2 F_{mn}F^{mn} + \frac{1}{24\pi G_4}\frac{\phi_{,m}\phi^{,m}}{\phi^2}\right) \tag{1.6}$$

Comparing (1.6) with (1.4), one sees that

$$G_4 = \frac{G_5}{2\pi L} \tag{1.7}$$

Although we started off to combine gravity with electromagnetism, we ended up with an additional coupling to the scalar field ϕ. Kaluza and Klein set $\phi = 1$ although they were not comfortable about it. Today, the existence of a scalar field is not a suspicious idea anymore.

1.3 Brane World Scenarios

The brane world scenarios started as a more realistic alternative to the Kaluza-Klein picture. In the Kaluza-Klein picture, for an effective four dimensional spacetime, the size of the extra dimensions must be microscopical. Common sense tells that this should be about the Planck length ($l_{\rm pl} \sim 10^{-33}$ cm) although there are works where the compactification was made at the length of the electroweak scale [10]. But there is another mechanism in which, the ordinary matter (*SM fields*) is trapped to a four dim

ensional submanifold (or a *membrane*) of the fundamental space [14, 15]. The size of the extra dimensions need not be small, in fact they may even be infinite, thus arising the possibility to observe the extra dimensions. That's one of the many reasons why there is so much interest towards the brane world scenarios: a potential detectability… But the most important feature is that the general idea of the lower dimensional manifolds (or *p-branes* in that context) is a natural consequence of the M-Theory. For example, gauge fields can be localized on D-branes [11]. Although some realistic brane world scenarios based on M-Theory had been proposed [12,13], most of the phenomenological models have nothing to do with M-theory's p-branes, but there is hope that there will be some counterparts in the fundamental theory. Hence, the term *brane* used in this work stands for any four dimensional hypersurface on which ordinary matter is localized, regardless of the mechanism by which it is trapped.

So somehow the matter fields (ie. The SM fields) are localized on a (3+1) dimensional membrane or domain wall, embedded in a (3+1+d) dimensional manifold. In the brane world picture, the extra dimensions may be large, even infinite. Although the idea of brane worlds goes back to the sixties [16], a detailed investigation was made only recently[2].

1.3.1 Arkani-Hamed, Dimopoulos and Dvali's (ADD) Brane World Model

The model introduced by Arkani-Hamed, Dimopoulos and Dvali (ADD) [18,19] was the first brane world scenario and intended to solve the hierarchy problem or to put it in another way, to explain why gravity is weaker than other forces. They started by neglecting the brane tension and by compactifying the extra dimensions. In a way, they reintroduced the KK picture, though the size of the extra dimensions L could be large. The SM fields may stop four dimensional behavior (according to dynamics on the brane) much below L, but gravitation becomes multi dimensional just below L. In the light of recent experiments which verified Newton's inverse square law at

[2] see note in [17] for a brief history of brane world scenarios

distances about 0.2 mm [22], one can say that the size of the extra dimensions must be as large as 0.1 mm [3].

This is an opportunity to attack the hierarchy problem. In a theory with d extra dimensions, the gravitational action can be written as

$$S = -\frac{1}{16\pi G_{4+d}} \int d^4x d^d z \sqrt{- {}^{(4+d)}g} \; {}^{(4+d)}R \tag{1.8}$$

where $G_{4+d} = 1/M^{d+2}$ is the (4+d) dimensional Newton's constant. Note that the multi-dimensional gravitational scale M is the fundamental mass parameter here, instead of the four dimensional Planck scale. In ADD model, the graviton zero mode mediates the four dimensional gravity, hence the wave function is homogeneous over the extra dimensions. This lets us take the metric independent of the extra coordinates. Doing the trivial z integration in (1.8), one gets

$$S = -\frac{M^{d+2}}{16\pi} V_d \int d^4x \sqrt{-h} \; {}^{(4)}R \tag{1.9}$$

where V_d is the volume of the extra dimensions. Now taking the four dimensional gravitational scale as the Planck scale, we can deduce from (1.9) that

$$M_{pl}^2 = M^{d+2} V_d \tag{1.10}$$

Now to understand what (1.10) means, one may take the size of all extra dimensions as L. Then from (1.10) one can express the size L in terms of the mass scales as

$$L = \frac{M_{pl}^{2/d}}{M^{1+2/d}} \tag{1.11}$$

If L is large compared to the fundamental length M^{-1}, the Planck mass should be much larger than the fundamental scale M. This is the breakthrough of the ADD model. Speculating that there should be only one fundamental scale, they took M to be the electroweak scale $\left(M_{EW} \sim 1 \text{ TeV}\right)$ so that the source of the hierarchy between

[3] Actually until the publishing of [18,19], the inverse square law was established at several milimeter

the two scale will be the largeness of the extra dimensions. Using this assumption in (1.11) one gets

$$L \sim \frac{\left(10^{19}\ \text{GeV}\right)^{2/d}}{\left(1\ \text{TeV}\right)^{1+2/d}} = 10^{-17+32/d}\ \text{cm} \tag{1.12}$$

From (1.12), one sees that if there is only one extra dimension, L has an impossible value of $L \sim 10^{15}$ cm, but for $d = 2$, L has a more acceptable value of about a milimeter though this distance is still in the range of Newtonian gravity. That is why astrophysicists and cosmologists exclude the scale $M \sim 1$ TeV for $d = 2$, but a more decent value of $M \sim 30$ TeV suggests extra dimensions with the size of one to ten micrometers. This is another motivation for the experimentalists to explore deviations from the inverse square law in micrometer range [23-27].

It should be noted that in a more realistic $d = 6$ choice, the size goes down to 10^{-12} cm which is still larger than the electroweak scale which is roughly 10^{-17} cm. However, one should keep in mind that those figures are obtained by assuming that all the extra dimensions are of the same size. If some dimensions are smaller than others then one may observe deviations from the inverse square law for $d > 2$.

1.3.2 Randall-Sundrum (RS) Brane World Models

1.3.2.1 Randall-Sundrum's First Brane World Model (RS1)

ADD's result is remarkable because it opened a door for low energy tests to check the existence of the extra dimensions. But by solving the hierarchy between gravity and other forces, they generated another hierarchy between the weak scale and the compactification scale [28].

$$\frac{1}{L^{1/d}} \ll M_{EW} \tag{1.13}$$

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range [20,21]. The idea of large extra dimensions inspired the research in [22].

Randall and Sundrum's first model (RS1) [28] was another attempt to solve the hierarchy problem. They started off with two domain walls with opposite brane tensions and between them they put an AdS_5 bulk (see figure 1.1).

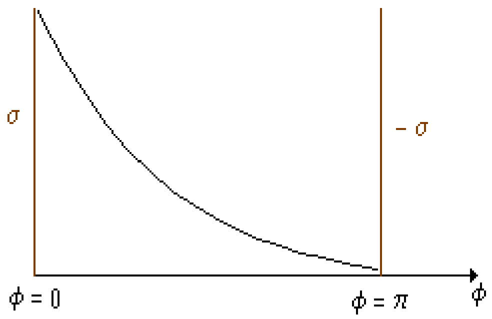

Figure 1.1 The shape of RS1 model. At ϕ = 0 (hidden brane) the warp factor is maximum, at $\phi = \pi$ (visible brane) it is minimum.

The action for this model is

$$\begin{aligned} S &= \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{-g}\left(-\Lambda + 2M^3 R\right) \\ &+ \int d^4x \sqrt{-h_{vis}}\left(\mathcal{L}_{vis} - V_{vis}\right) + \int d^4x \sqrt{-h_{hid}}\left(\mathcal{L}_{hid} - V_{hid}\right) \end{aligned} \tag{1.14}$$

Where g is the five dimensional metric, h_{vis} and h_{hid} are the four dimensional metrics on the branes and V_{vis} and V_{hid} are constant "vacuum energy" which act as a gravitational source. The first term of (1.14) is the five dimensional Einstein-Hilbert action while the others correspond to visible and hidden branes.

The four dimensional metrics on the branes are defined as

$$\begin{aligned} h_{ij}^{hid}\left(x^i\right) &\equiv g_{AB}\left(x^i, \phi = 0\right) \\ h_{ij}^{vis}\left(x^i\right) &\equiv g_{AB}\left(x^i, \phi = \pi\right) \end{aligned} \tag{1.15}$$

By minimizing the action of (1.14), one can find the five dimensional field equations

$$\begin{aligned} \sqrt{-g}\left(R_{AB} - \frac{1}{2} g_{AB} R\right) = -\frac{1}{4M^3}\{&\Lambda\sqrt{-g}G_{AB} + V_{vis}\sqrt{-h_{vis}}h_{ij}^{vis}\delta_A^i\delta_B^j\delta\left(\phi - \pi\right) \\ &+ V_{hid}\sqrt{-h_{hid}}h_{ij}^{hid}\delta_A^i\delta_B^j\delta\left(\phi\right)\} \end{aligned} \tag{1.16}$$

The form of the five dimensional metric can be found by requiring four dimensional Poincaré invariance on the brane. The metric ansatz authors of [28] chose is

$$ds^2 = e^{-2\sigma\phi}\eta_{ij}dx^i dx^j + r_c^2 d\phi^2 \tag{1.17}$$

where x^j are the usual four dimensional coordinates and ϕ is the fifth coordinate which is a finite interval determined by r_c. Using this ansatz in the field equations (1.16) one gets

$$\frac{6\sigma'^2}{r_c^2} = -\frac{\Lambda}{4M^3} \tag{1.18}$$

$$\frac{3\sigma''}{r_c^2} = \frac{V_{hid}}{4M^3 r_c}\delta(\phi) + \frac{V_{vis}}{4M^3 r_c}\delta(\phi - \pi) \tag{1.19}$$

Solving (1.18) and requiring orbifold symmetry $\phi \to -\phi$ yields

$$\sigma = r_c|\phi|\sqrt{-\frac{\Lambda}{24M^3}} \tag{1.20}$$

For (1.20) to have a physical meaning, the cosmological constant must be negative, a not surprising outcome of AdS_5 bulk.

By considering the metric a periodic function in ϕ and derivating (1.20) twice, one can deduce

$$\begin{aligned} V_{hid} &= -V_{vis} = 24M^3 k \\ \Lambda &= -24M^3 k \end{aligned} \tag{1.21}$$

where k is a scale of the order the Planck scale. The final form of the RS1 metric is then

$$ds^2 = e^{-2kr_c|\phi|}\eta_{ij}dx^i dx^j + r_c^2 d\phi^2 \tag{1.22}$$

The model introduced, one can attempt to attack the hierarchy problem. Note that in (1.22) the metrics on the branes are both conformally flat. Let us consider metric perturbations γ_{ij} to the flat metric

$$h_{ij} = \eta_{ij} + \gamma_{ij} \tag{1.23}$$

Using (1.23) in the gravitational action and integrating over the fifth coordinate one can obtain the four dimensional effective action:

$$\begin{aligned} S_{eff} &= 2M^3 \int dx^4 \sqrt{-hr_c^2}\,\overline{R} \int_{-\pi}^{\pi} d\phi e^{-2kr_c|\phi|} + \ldots \\ &= \frac{2M^3}{k}\left(1 - e^{-2kr_c\pi}\right) \int dx^4 \sqrt{-h}\,\overline{R} + \ldots \end{aligned} \tag{1.24}$$

where $\overline{R}$ is the Ricci scalar defined by the metric h_{ij}. As the four dimensional effective scale for gravity is of the order the Planck scale, from equation (1.24), one can write down:

$$M_{pl}^2 = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right) \tag{1.25}$$

(1.25) shows that even for large kr, the Planck scale depends only weakly to the five dimensional gravitational scale. But for the physical masses of SM, things are different.

Consider a Higgs field localized on the visible brane, the action will be:

$$S_{vis} = \int d^4x \sqrt{-h_{vis}} \left(h_{vis}^{ij} D_i H^\dagger D_j H - \lambda \left(|H|^2 - m_0^2 \right)^2 \right) + \ldots \tag{1.26}$$

where m_0 is the five dimensional mass parameter.

As the visible brane is located at $\phi = \pi$, the metric on the brane will be

$$g_{ij}\left(x^i, \phi = \pi\right) = h_{ij} e^{-2kr_c\pi} \tag{1.27}$$

Combining equation (1.26) with (1.27) and renormalizing the wave function as $H \to e^{kr_c\pi} H$, one gets

$$S_{vis} = \int d^4x\sqrt{-h}\left(h^{ij} D_i H^\dagger D_j H - \lambda\left(|H|^2 - e^{-2kr_c\pi} m_0^2\right)^2 \right) + \ldots \tag{1.28}$$

The result is not limited to a Higgs field. Generally the effective mass on the brane is related to the five dimensional mass as:

$$m = e^{-kr_c\pi} m_0 \tag{1.29}$$

So, supposing that the bare Higgs mass is of order the Planck scale (~ 10^{19} GeV), to be able to observe the mass on the brane at the EW scale (~ 1 TeV), we only need to set kr_c ~ 12. This way, there is no very large hierarchy between the fundamental parameters.

In the argument above, we have set the fundamental scale to the Planck scale, but this is not mandatory. Consider a coordinate transformation $x^i \to e^{kr_c\pi} x^i$. Using this in the metric of (1.17) then in (1.26), we see that no rescaling occurs: the Higgs mass observed on the brane is equal to its physical mass. But the gravitational scale is different now. Applying the transformation to the effective gravitational action of (1.24), one sees that the Planck scale is

$$M_{pl}^2 = \frac{M^3}{k}\left(e^{2kr_c\pi} - 1\right) \tag{1.30}$$

This time, the fundamental scale is set to be the EW scale. Again, there is no additional hierarchy, the scale of kr_c is of the same order as before. This way, Randall and Sundrum not only solved the hierarchy problem, their result tells us that any scale can be fundamental. Also, the non-existence of any extra hierarchy between the fundamental parameters raises the importance of their work.

1.3.2.2 Randall-Sundrum's Second Brane World Model (RS2)

The brane world scenario of Section 1.3.2.1 leads to some unexpected results. If the gravitational scale is of order the electroweak scale, we should be able to observe a five dimensional gravity, though experimentally we know that gravity looks four dimensional up to two tenths of a milimeter [22]. Although one can localize gravity by compactifying the extra dimensions, the RS2 model tackles this problem with an infinite extra dimension and the solution reproduces Newton's inverse square law on the brane [29].

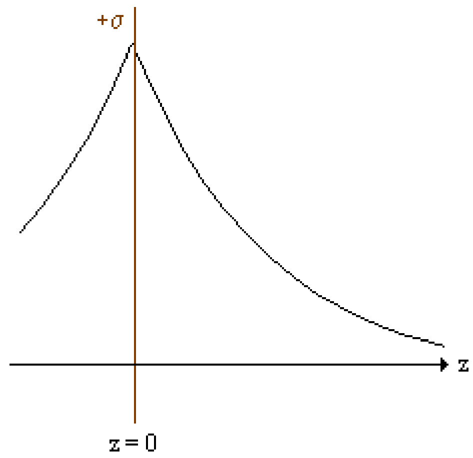

Figure 1.2 The shape of RS2 model. Note that there is only one brane with a positive tension, located at $z = 0$.

The set-up is a little different than the preceding one. In RS1 model, the visible brane (our world) had negative tension. This ensured an exponential term in scale equations (1.25) and (1.29). But as the visible brane was at the warp factor's minimum, gravity was localized on the hidden one. Forgetting about the hierarchy problem, one may concentrate on localizing gravity on the brane. So this time, the positive tension brane is the visible one and the other is brought to infinity. Taking the metric (1.17) with $r_c \to \infty$ one gets

$$ds^2 = e^{-2k|z|}\eta_{ij}dx^i dx^j + dz^2 \tag{1.31}$$

By minimizing the gravitational action, and considering low energy approximations, the authors of [29] found the KK spectrum of the effective four dimensional theory and finally deduced a three dimensional gravitational potential due to a point mass m on the brane as

$$\phi(r) = G_N \frac{m}{r}\left(1 + \frac{1}{r^2 k^2}\right) \tag{1.32}$$

(1.32) is the Newtonian potential of four dimensional gravity with Yukawa type corrections at distances $r < k^{-1}$ (short distances).

The RS2 model destroyed the prejudice (or *lore* as the authors stated [29]) regarding the compactness and size of the extra dimension. It opened a whole new perspective on multi-dimensionality and on our place in it.

1.3.3 Some Other Brane World Models

In Sections 1.4.1 - 1.4.3, we reviewed the most important brane world models though there are two more we should briefly introduce.

The first of these models is an attempt to unify the approach to the hierarchy problem in RS1 and the localization of gravity on brane in RS2 [30].

In this case, there are two positive non-equal tension branes. The one with larger tension is called *The Planck Brane*, and the other, *The TeV Brane*. Although ordinary matter is localized on the TeV brane, gravity is effectively four dimensional in both. The hierarchy problem is solved just as in [28] when the visible brane is taken as the TeV brane.

The second model that will be discussed here is actually a collection of models. In the RS models, the brane cosmological constants are set to zero by (1.21) resulting a flat metric on the brane. Because of that, those models are called *critical brane worlds*. But the recent studies of supernovae claim that the cosmological constant must be small but positive [32,33]. The unrealistic criticality condition (1.21) is a consequence of the metric ansatz (1.17), which can be generalized to allow de Sitter and anti-de Sitter branes [31]

$$ds^2 = a^2(z)\, h_{ij} dx^i dx^j + dz^2 \tag{1.33}$$

The action for this model is given by

$$S = \int d^4xdz\sqrt{-g}\left(-\frac{1}{4}R - \Lambda_{(5)}\right) - \lambda\int d^4xdz\sqrt{-h}\delta(z) \tag{1.34}$$

where λ is the four dimensional cosmological constant on the brane. The bulk cosmological constant is taken as $\Lambda = -6k^2$ [28]. So the solution will satisfy the boundary conditions on a positive tension brane. The five dimensional Einstein equations read [34-37]

$$\frac{\lambda}{a^2} - 3\left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a} = -4k^2 \tag{1.35}$$

$$-4\frac{\ddot{a}}{a} = -4k^2 \tag{1.36}$$

where an overdot denotes differentiation with respect to the fifth coordinate. According to the sign of λ, the solutions of (1.35) and (1.36) will be

$\lambda > 0$, $$a(z) = \frac{1}{k}\sqrt{\frac{\lambda}{3}}\sinh(\pm kz + c) \tag{1.37}$$

$\lambda = 0$, $$a(z) = e^{\pm kz + c} \tag{1.38}$$

$\lambda < 0$, $$a(z) = \frac{1}{k}\sqrt{-\frac{\lambda}{3}}\cosh(\pm kz + c) \tag{1.39}$$

where c is a constant of integration.

Imposing Israel junction conditions [38] (see also Chapter 4)

$$\Delta K_{ij} = -\frac{8\pi G_5}{3}\sigma h_{ij} \tag{1.40}$$

where K_{ij} is the extrinsic curvature and will be defined in Chapter 3.

Using the general ansatz of (1.33) and imposing $\mathbb{Z}_2$ symmetry across the brane one finds

$$\left.\frac{\dot{a}}{a}\right|_{z\to 0^+} = -\frac{4\pi G_5}{3}\sigma \tag{1.41}$$

As we assumed a positive tension, equations (1.37), (1.38) and (1.39) will become

$$\lambda > 0\,, \qquad a(z) = -\frac{1}{k}\sqrt{\frac{\lambda}{3}}\sinh(kz - c) \tag{1.42}$$

$$\lambda = 0\,, \qquad a(z) = e^{-kz+c} \tag{1.43}$$

$$\lambda < 0\,, \qquad a(z) = \frac{1}{k}\sqrt{-\frac{\lambda}{3}}\cosh(kz - c) \tag{1.44}$$

The brane tensions for each case will be

$$\lambda > 0\,, \qquad \tilde{\sigma} = k\coth c \ > k \tag{1.45}$$

$$\lambda = 0\,, \qquad \tilde{\sigma} = k \tag{1.46}$$

$$\lambda < 0\,, \qquad \tilde{\sigma} = k\tanh c \ < k \tag{1.47}$$

where $\tilde{\sigma} = 4\pi G_5\sigma/3$. If one demands that the factor a to be equal to one on the brane, one gets

$$\lambda > 0\,, \qquad k = \sqrt{\frac{\lambda}{3}}\sinh c \tag{1.48}$$

$$\lambda = 0\,, \qquad c = 0 \tag{1.49}$$

$$\lambda < 0 \,, \qquad k = \sqrt{-\frac{\lambda}{3}} \cosh c \tag{1.50}$$

Comparing (1.48) and (1.50) with (1.45) and (1.47), one can deduce the brane cosmological constants for the non-critical cases as

$$\lambda = 3\left(\tilde{\sigma}^2 - k^2\right) \tag{1.51}$$

The final solutions are summarized in the table below.

Table 1.1 A summary of the non-critical and critical brane-world model solutions. Here, σ_c denotes the tension of the critical brane.

Model	**Tension**	**Induced Metric**	**The *a*(*z*) factor**	***k***	**Cosmological Constant**
RS2	$\sigma = \sigma_c$	Minkowski $\left(\eta_{ij}\right)$	$e^{-k\lvert z\rvert}$	-	$\lambda = 0$
Sub-critical	$\sigma < \sigma_c$	AdS_4	$\frac{1}{k}\sqrt{-\frac{\lambda}{3}} \cosh\left(c - k\lvert z\rvert\right)$	$\sqrt{-\frac{\lambda}{3}} \cosh c$	$\lambda < 0$
Super-critical	$\sigma > \sigma_c$	dS_4	$\frac{1}{k}\sqrt{\frac{\lambda}{3}} \sinh\left(c - k\lvert z\rvert\right)$	$\sqrt{\frac{\lambda}{3}} \sinh c$	$\lambda > 0$

The study of the non-critical branes, and especially of super-critical branes may provide solutions with effective four dimensional cosmological constant with correct sign. Although the induced cosmological constant is negative, AdS_4 brane in a AdS_5 bulk is also interesting and has been investigated [31] for implications for holography.

2. A BRIEF REVIEW ON BRANE WORLD GRAVITY

The result of [28], where gravity is localized on the brane, made a real impact on theoretical physicists and a search began for answers on what gravitational fields, due to sources on the brane, look like, on and off the brane.. Here, we will briefly skim over some important studies on black holes and on four dimensional effective gravity.

At first sight, to acquire a black hole on the brane in RS2 model may seem straightforward. Moreover, replacing the four dimensional flat metric in (1.31) with any Ricci flat metric, the five dimensional field equations are still satisfied[4]. The first thing that comes in mind then is to replace η_{ij} with the Schwarzchild solution, thus on the brane, one observes a Schwarzschild black hole, which is in five dimensions a *black string* in AdS. However, those solutions have *Gregory-Laflamme instability* [43] near the AdS horizon. The first RS2-based black hole solution was a generalization of these black holes and was called the *black cigar* solution [44]. When this *cigar* extended all the way down to the AdS horizon, the metric for the black string was recovered. But the authors walked around the instability by conceiving that their solution, far from the AdS horizon, looks like a black string, but has its horizon closed off before reaching the AdS horizon. Their conclusions were supported by the works of Emparan, Horovitz and Myers, where they calculated exact black hole solutions, stationary [45] and rotating [46], on a 2-brane by considering an AdS-C bulk.

There is a number of work on linearized gravity in brane backgrounds, where the solution of RS2 model (1.32) is inspected more thoroughly and the additions to the Newton's Law in order of r^{-3} is verified in different approaches [47-50]. Moreover, there has been works on linearized gravity on alternative backgrounds [51] and Karch-Randall backgrounds [52] among others, enlarging our knowledge on the aspects of the gravity on the brane at long distances. But the most fruitful work was due to Shiromizu, Maeda and Sasaki [42] who wrote down the effective field equations for a 3-brane (see Chapter 5) and recovered Einstein equations for long distances, using *Gaussian normal coordinates* where an *acceleration-free* condition is satisfied

$$n^A \nabla_A n^B = 0 \tag{2.1}$$

where n^A is the unit normal vector to the brane. Using their equations, a solution for a static spherically symmetrical black hole on the brane was established. Interestingly, in the absence of a Maxwell field, the solution resembled the Reissner-Nordström solution [53]. A more detailed study, again using the equations of [42] showed that, in the presence of a gauge field on the brane, the induced metric is Reissner-Nordström with two types of charges [54]. Assuming a static, spherically symmetric metric in form

$$ds^2 = -U(r)\,dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega_2^2 \tag{2.2}$$

and using it in the 3-brane field equations, they obtained

$$U(r) = 1 - \frac{2G_4 M}{r} + \frac{\beta + Q^2}{r^2} + \frac{l^2 Q^4}{20 r^6} \tag{2.3}$$

where Q is the Maxwell charge and β is the tidal charge resulting from the nonlocal bulk effects.

Although the metric form of a static black hole on the brane is now known, there's still not much knowledge on what the bulk metric for this case looks like. Linearized gravity investigations, suggest that a brane-world black hole should be "pancake" shaped, smoothly going to the bulk then back on the brane. The solution of [54] has curvature singularities at a finite distances in the extra dimensions.

In the following chapters, we will be constructing effective field equations in a general coordinate system. This way, we show that some extra terms that may

[4] This is discussed for general p-brane solutions in [55]

describe some shaded effects of the bulk (such as acceleration in the fifth dimension) in [42] will reveal. We will then find a general solution to the Hamiltonian constraint equation describing a rotating black hole on the brane.

3. FOILATION OF THE FIVE DIMENSIONAL MANIFOLD INTO FOUR DIMENSIONAL TIME-LIKE HYPERSURFACES

As a model of a brane world embedded in five dimensional bulk, we chose the *spacelike slicing* method used in Hamiltonian formulation of GR (see [39-41]) in a coordinate setting in the fashion of Arnowitt, Deser and Misner [56].

We begin with a five dimensional manifold M_5 . On M_5 we introduce coordinate system x^A with $A = 0, 1, 2, 3, 4$. Let the line element on M_5 be

$$ds^2 = g_{AB} dx^A dx^B \tag{3.1}$$

Next, we consider a foilation of the five dimensional spacetime into a family of non-intersecting time-like hypersurfaces defined by

$$z\left(x^A\right) = \text{Constant} \tag{3.2}$$

3.1 Coordinates on the Hypersurface

Now, we need to define a coordinate system on Σ_z . A priori, the coordinates on Σ_z need not coincide with those of $\Sigma_{z'}$. It is, however, convenient to introduce a relationship as follows.

Consider a congruence of curves γ intersecting the hypersurfaces Σ_z (see Fig. 2.1.1). Let (P, Q, R) , (P', Q', R') and (P'', Q'', R'') be events on hypersurfaces Σ_z , $\Sigma_{z'}$, $\Sigma_{z''}$ respectively. Then we may define a coordinate system (y^i , $i = 0, 1, 2, 3$) on Σ_z such that

$$y^i(P) = y^i(P') = y^i(P'') \tag{3.3}$$

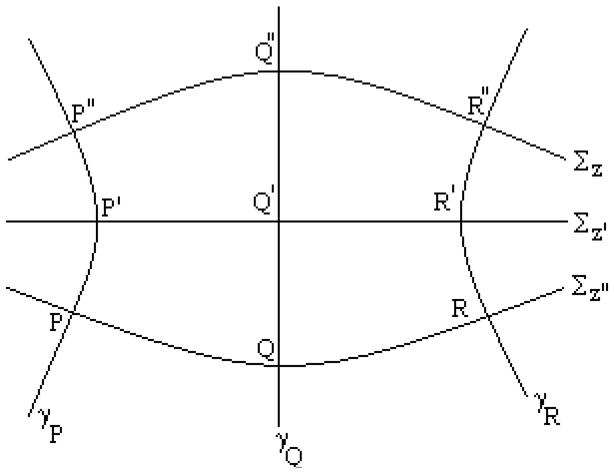

Figure 3.1 A two dimensional example of a hypersurface family Σ_z on M_5 . The curves γ connect the hypersurfaces. Note that the curves do not have to intersect the surfaces orhogonally.

This construction defines another coordinate system (z, y^i)on M_5. There exists a transformation between the two coordinates

$$x^A = x^A\left(y^i, z\right) \tag{3.4}$$

The vector tangent to the curves $\gamma\left(y^i\right)$

$$Z^A = \left(\frac{\partial x^A}{\partial z}\right)_{y^i} \tag{3.5}$$

is called the *evolution vector*. The vectors tangent to hypersurfaces Σ_z are

$$e_i^A = \left(\frac{\partial x^A}{\partial y^i}\right)_z \tag{3.6}$$

3.2 Unit Normal Vector to the Hypersurface

The tangent vectors established, we must define a vector normal to Σ_z ie. $n^A n_A = 1$. As the value of z changes only on the direction orthogonal to Σ_z , covector $\partial_A z$ is indeed normal to that hypersurface. The unit normal to Σ_z is then

$$n_A = \frac{\partial_A z}{\sqrt{\left|\left(\partial_A z\right)\left(\partial^A z\right)\right|}} = N\partial_A z \tag{3.7}$$

where the scalar *N* defined by $N = \left|\left(\partial_A z\right)\left(\partial^A z\right)\right|^{-1/2}$ is called the *lapse function*. It is straightforward to see that

$$e_i^A n_A = 0 \tag{3.8}$$

3.3 Induced Metric on the Hypersurface

Since the curves γ do not generally intersect Σ_z orthogonally, Z^A is not necessarily parallel to n^A . Therefore, we can decompose Z^A in the basis of normal and tangent vectors. For the normal part, we make use of (3.5) and (3.7)

$$Z^A n_A = \left(\frac{\partial x^A}{\partial z}\right)\left(N\frac{\partial z}{\partial x^A}\right) = N \tag{3.9}$$

and the tangent part is defined as

$$Z^A e_A^i = N^i \tag{3.10}$$

where the four-vector N^i is called the *shift vector*.

$$Z^A = Nn^A + N^i e_i^A \tag{3.11}$$

The displacement on hypersurface Σ_z can be calculated using (3.1) for constant *z*

$$\begin{aligned} ds_\Sigma^2 &= \left(g_{AB}dx^A dx^B\right)_z \\ &= g_{AB}\left(\frac{\partial x^A}{\partial y^i}dy^i\right)\left(\frac{\partial x^B}{\partial y^j}dy^j\right) \\ &= h_{ij}dy^i dy^j \end{aligned} \tag{3.12}$$

where $h_{ij} = g_{AB} e_i^A e_j^B$ is the induced metric (or "first fundamental form") on Σ_z.

3.4 The Metric in Coordinate System (y^i, z)

From (3.12), we infer that

$$g_{AB} = n_A n_B + h_{ij} e_A^i e_B^j \tag{3.13}$$

Using (3.7), we can write down

$$n_A = \left(0,0,0,0,N\right) \tag{3.14}$$

To be able to express the five dimensional metric in coordinates (y^i , z), we first express dx^A in that coordinate as

$$\begin{aligned} dx^A &= \frac{\partial x^A}{\partial z} dz + \frac{\partial x^A}{\partial y^i} dy^i \\ &= Z^A dz + e_i^A dy^i \\ &= \left(Nn^A + e_i^A N^i\right) dz + e_i^A dy^i \\ &= \left(Ndz\right) n^A + \left(dy^i + N^i dz\right) e_i^A \end{aligned} \tag{3.15}$$

Now we use the expression (3.15) in (3.1) to express the line element in coordinates (y^i , z)

$$\begin{aligned} ds^2 &= g_{AB} \left(\left(Ndz\right) n^A + \left(dy^i + N^i dz\right) e_i^A\right)\left(\left(Ndz\right) n^B + \left(dy^j + N^j dz\right) e_j^B\right) \\ &= h_{ij} \left(dy^i + N^i dz\right)\left(dy^j + N^j dz\right) + N^2 dz^2 \\ &= h_{ij} dy^i dy^j + 2N_i dy^i dz + \left(N^2 + N^i N_i\right) dz^2 \end{aligned} \tag{3.16}$$

We can express (3.16) in matrix form as

$$g_{AB} = \begin{pmatrix} h_{ij} & N_i \\ & \\ N_j & N^2 + N_i N^i \end{pmatrix} \tag{3.17}$$

The inverse of (3.17) is then

$$g^{AB} = \begin{pmatrix} h^{ij} + \dfrac{N^i N^j}{N^2} & -\dfrac{N^i}{N^2} \\ -\dfrac{N^j}{N^2} & \dfrac{1}{N^2} \end{pmatrix} \tag{3.18}$$

Now raising (3.14) with (3.18), we find the components of the unit normal in contravariant form as

$$n^A = \left(-\frac{N^i}{N}, \frac{1}{N} \right) \tag{3.19}$$

3.5 Extrinsic Curvature

We introduce the 4-tensor

$$\begin{aligned} K_{ij} &= \nabla_{(A} n_{B)} e_i^A e_j^B \\ &= \frac{1}{2}\left(\nabla_A n_B + \nabla_B n_A \right) \\ &= \frac{1}{2}\left(\pounds_n g_{AB} \right) e_i^A e_j^B \end{aligned} \tag{3.20}$$

called the *extrinsic curvature* or *the second fundamental form* of the hypersurface. It describes the extrinsic aspect of the spacetime: the embedding of the hypersurface in the enveloping spacetime manifold. Using (3.11), we find

$$K_{ij} = \frac{1}{2N} e_i^A e_j^B \left(\nabla_A Z_B + \nabla_B Z_A - \nabla_B N_A - \nabla_A N_B \right) \tag{3.21}$$

where we defined $N_A = N_i e_A^i$. Projecting the derivatives of the shift vector on the hypersurface and defining a covariant derivative operator D_i compatible with the induced metric h_{ij}, we get

$$K_{ij} = \frac{1}{2N}\left(\left(\mathfrak{L}_Z g_{AB}\right) e_i^A e_j^B - D_i N_j - D_j N_i\right) \tag{3.22}$$

In a particular coordinate system where

$$\begin{aligned} x^i &= y^i \ , \ \left(i = 0,1,2,3\right) \\ x^5 &= z \end{aligned} \tag{3.23}$$

the expression (3.22) reduces to

$$K_{ij} = \frac{1}{2N}\left(\frac{\partial h_{ij}}{\partial z} - D_i N_j - D_j N_i\right) \tag{3.24}$$

From now on, we will use the coordinate system of (3.23).

3.6 The Metric Connections on the Hypersurface

We introduce the metric connections on the hypersurface as

$$\lambda^i_{jk} \equiv \frac{1}{2} h^{im}\left(\partial_j h_{mk} + \partial_k h_{jm} - \partial_m h_{jk}\right) \tag{3.25}$$

To be able to calculate the g_{AB} connections in terms of the hypersurface quantities, let's first define those

$$\Gamma^A_{BC} \equiv \frac{1}{2} g^{AM}\left(\partial_B g_{MC} + \partial_C g_{BM} - \partial_M g_{BC}\right) \tag{3.26}$$

Using the metric expression (3.16) in definition (3.25), we calculated the five dimensional connections as

$$\Gamma^i_{jk} = \lambda^i_{jk} + \frac{N^i}{N} K_{jk} \tag{3.27}$$

$$\Gamma^5_{ij} = -\frac{1}{N} K_{ij} \tag{3.28}$$

$$\Gamma^5_{5i} = N^m \Gamma^5_{mi} + D_i f \tag{3.29}$$

$$\Gamma^5_{55} = N^m \Gamma^5_{5m} + \partial_5 f \tag{3.30}$$

$$\Gamma^i_{5j} = -N^i \Gamma^5_{5j} + N K^i_j + D_j N^i \tag{3.31}$$

$$\Gamma^i_{55} = -N^i \Gamma^5_{55} + \partial_5 N^i + 2N K^i_m N^m - N^2 D^i f + N^m D_m N^i \tag{3.32}$$

where $f \equiv \log(N)$. Note that we have used the index *5* to denote the fifth dimension.

3.7 Riemann Tensor on the Hypersurface

We introduce the Riemann curvature four-tensor intrinsic to the hypersurface in terms of the h_{ij} connections

$$^{(4)}R^i_{jkl} \equiv \partial_k \lambda^i_{jl} - \partial_l \lambda^i_{jk} + \lambda^m_{jl} \lambda^i_{km} - \lambda^m_{jk} \lambda^i_{ml} \tag{3.33}$$

whereas the five dimensional Riemann five-tensor is defined as

$$R^A_{BCD} \equiv \partial_C \Gamma^A_{BD} - \partial_D \Gamma^A_{BC} + \Gamma^M_{BD} \Gamma^A_{CM} - \Gamma^M_{BC} \Gamma^A_{MD} \tag{3.34}$$

Using the expressions (3.27)-(3.32), we calculated the five dimensional Riemann tensors in terms of the four dimensional quantities

$$R_{ijkl} = {}^{(4)}R_{ijkl} + K_{il} K_{jk} - K_{ik} K_{jl} \tag{3.35}$$

$$R_{5ijk} = N^m R_{mijk} + N\left(D_k K_{ij} - D_j K_{ik}\right) \tag{3.36}$$

$$R_{5i5j} = N^m R_{5imj} + N\left(N^m D_i K_{jm} - \partial_5 K_{ij} + K_{im} D_j N^m + K_{mj} D_i N^m\right) \\ + N^2\left(K_i^m K_{mj} - D_j D_i f - D_j f D_i f\right) \tag{3.37}$$

3.8 Ricci Tensor on the Hypersurface

The Ricci tensor intrinsic to the hypersurface Σ_z is defined as

$$^{(4)}R_{ij} \equiv h^{mn}\ ^{(4)}R_{imjn} \tag{3.38}$$

Contracting the Riemann tensors of (3.35)-(3.37) with the metric (3.18), we express the five dimensional Ricci four-tensor as

$$R_{ij} = \ ^{(4)}R_{ij} - D_i D_j f - D_i f D_j f + \frac{1}{N}\mathfrak{L}_{\vec{N}} K_{ij} - \frac{1}{N}\partial_5 K_{ij} + 2K_i^m K_{mj} - KK_{ij} \tag{3.39}$$

$$R_{i5} = N^m R_{im} + N\left(D_m K_i^m - D_i K\right) \tag{3.40}$$

$$R_{55} = N^m R_{m5} - N^2\left(D_m D^m f + D_m f D^m f\right) - N^2 K_{mn} K^{mn} + N\left(N^m D_n K_m^n - \partial_5 K\right) \tag{3.41}$$

Where $\mathfrak{L}_{\vec{N}}$ is the Lie derivative with respect to the shift vector and K is defined as $K \equiv K_m^m$.

3.9 Scalar Curvature on the Hypersurface

The intrinsic scalar curvature is defined as

$$^{(4)}R \equiv h^{mn}\ ^{(4)}R_{mn} \tag{3.42}$$

Tracing the expressions (3.39)-(3.41), we calculated the five dimensional scalar curvatures in terms of the hypersurface quantities as

$$R = {}^{(4)}R - K^2 - K_{mn}K^{mn} - 2\left(D^m D_m f + D_m f D^m f\right) + \frac{2}{N}\left(\mathfrak{L}_{\vec{N}} - \partial_5 K\right) \qquad (3.43)$$

3.10 Einstein Tensor on the hypersurface

To be able to write down the field equations, we will introduce the four dimensional Einstein four-tensors

$$^{(4)}G_{ij} \equiv {}^{(4)}R_{ij} - \frac{1}{2}h_{ij}{}^{(4)}R \qquad (3.44)$$

The five dimensional Einstein tensor components are calculated as

$$\begin{aligned} G_{ij} = {}& {}^{(4)}G_{ij} + \frac{1}{N}\mathfrak{L}_{\vec{N}}\left(K_{ij} - h_{ij}K\right) - \frac{1}{N}\partial_5\left(K_{ij} - h_{ij}K\right) - 3KK_{ij} - \left(D_i D_j f + D_i f D_j f\right) \\ & + 2K_i^m K_{jm} + \frac{1}{2}h_{ij}\left[K^2 + K_{mn}K^{mn} + 2\left(D_m D^m f + D_m f D^m f\right)\right] \end{aligned} \qquad (3.45)$$

$$G_{i5} = N^m G_{im} + N\left(D_m K_i^m - D_i K\right) \qquad (3.46)$$

$$G_{55} = N^m G_{m5} - N\left(\mathfrak{L}_{\vec{N}}K - N^m D_n K_m^n\right) + \frac{1}{2}N^2\left(K^2 - {}^{(4)}R - K_{mn}K^{mn}\right) \qquad (3.47)$$

4. BOUNDARY CONDITIONS

In Chapter 3, we took the fifth coordinate z continous, the hypersurfaces Σ_z are infinitesimally thin in this approach. In order to write down the effective field equations on the brane, we must first make sure that physics is continuous there, or interpret any jump that occurs. The boundary conditions for thin hypersurfaces is due to Israel [38] and are called *Israel junction conditions*. The formalism adopted here is due to Poisson [41].

4.1 The Set-Up

Let Σ_z partition five dimensional spacetime into two regions M_5^+ and M_5^- . Let's call the metric on $M_5^\pm$ as $g_{AB}^\pm$ and coordinates as $x^{A\pm}$. The problem is to patch the two parts smoothly on Σ_z so that the union of metrics $g_{AB}^\pm$ forms a valid solutions of gravitational field equations.

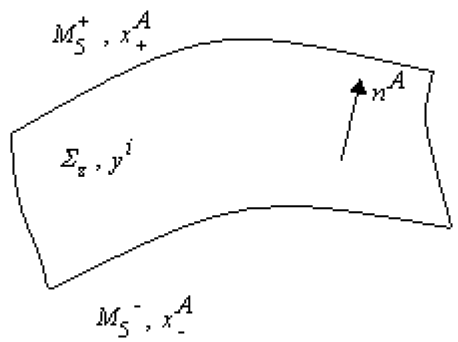

Figure 4.1 The hypersurface Σ_z as the boundary between two manifolds M_5^+ and M_5^-. Note that the normal vector's direction is chosen to point towards M_5^+.

Let the coordinates y^i on the two sides of Σ_z be same, be the coordinates on both sides of the hypersurface. Let's also choose the unit normal to point from M_5^- to M_5^+. Supposing that in the overlapping region there is a coordinate system x^A defined in the neighbourhood of Σ_z which is let pierced orthogonally by a congruence of geodesics that intersect it orthogonally. Let also the proper distance along the geodesic l vanish at the point of intersection. The normal vector is then[5]

$$n^A = \frac{\partial x^A}{\partial l} \tag{4.1}$$

Let the jump across Σ_z of any tensorial quantity A defined in both $M_5^{\pm}$ be

$$[A] = A\left(M_5^{\ +}\right)\Big|_{\Sigma} - A\left(M_5^{\ -}\right)\Big|_{\Sigma} \tag{4.2}$$

As l and x^A are continous across Σ_z , the jump of n_A is according to (4.1) is

$$\left[n^A\right] = 0 \tag{4.3}$$

Also, as y^i is the same on both sides of Σ_z , the jumps of the tangent vectors on the hypersurface according to (3.6) are

$$\left[e_i^A\right] = 0 \tag{4.4}$$

We will use the language of distribution in the fashion of [41]. We introduce the Heaviside distribution $\Theta(\ l\ )$ defined as

$$\Theta(l) = \begin{cases} +1\ ,\ \text{for } l > 0 \\ 0\ ,\ \text{for } l < 0 \\ \text{indeterminate}\ ,\ \text{for } l = 0 \end{cases} \tag{4.5}$$

and have the following properties

$$\Theta(l)\Theta(-l) = 0 \tag{4.6}$$

$$\Theta^2(l) = \Theta(l) \tag{4.7}$$

---

[5] This is actually the acceleration-free condition (2.1). This condition holds at least in the neighbourhood of the hypersurface [42].

$$\frac{d}{dl}\Theta(l) = \delta(l) \tag{4.8}$$

where $\delta(l)$ is the *Dirac distribution*. Also note that the product $\Theta(l)\delta(l)$ is not defined as a distribution.

4.2 The First Junction Condition

The definitions done, we begin by expressing the metric g_{AB} in the overlapping coordinates as a distribution valued tensor

$$g_{AB} = \Theta(l)\, g_{AB}{}^{+} + \Theta(-l)\, g_{AB}{}^{-} \tag{4.9}$$

From (4.9) we will go as far as Einstein tensor in the coordinates x^A. The next step on this track is expressing the connections in the overlapping region, for which we first calculate

$$\begin{aligned} \partial_C g_{AB} &= \Theta(l)\partial_C g_{AB}{}^{+} + \Theta(-l)\partial_C g_{AB}{}^{-} + \left(\delta(l)\, g_{AB}{}^{+} + \delta(-l)\, g_{AB}{}^{-}\right) n_C \\ &= \Theta(l)\partial_C g_{AB}{}^{+} + \Theta(-l)\partial_C g_{AB}{}^{-} + \delta(l)\, n_C \left[g_{AB}\right] \end{aligned} \tag{4.10}$$

in which, we used (4.1) and (4.8). Note that the last term in (4.10) is singular and the Christoffels will be problematic for they will have terms proportional to $\Theta(l)\delta(l)$. To be able to eliminate it, we must impose $\left[g_{AB}\right] = 0$, defined only in the overlapping coordinates. To generalize this, we will make use of (4.4)

$$\begin{aligned} &[g_{AB}]e_i^A e_j^B = 0 \\ &[h_{ij}] = 0 \end{aligned} \tag{4.11}$$

The coordinate independent expression (4.11) is the *first junction condition* and is a consequence of a well-defined geometry of the hypersurface.

4.3 The Second Junction Condition

In the preceding section, we set the metric continuous in the overlapping coordinates. We will see what conditions must one satisfy in order to get a continuous curvature across Σ_z . Now, using (4.10), one can express the connections in the x^A coordinates as

$$\Gamma^A_{BC} = \Theta(l)\Gamma^{A+}_{BC} + \Theta(-l)\Gamma^{A-}_{BC} \tag{4.12}$$

To express the Riemann tensors, we first have to derivate (4.12)

$$\begin{aligned}\partial_D\Gamma^A_{BC} &= \delta(l)\Gamma^{A+}_{BC}n_D + \Theta(l)\partial_D\Gamma^{A+}_{BC} - \delta(l)n_D\Gamma^{A-}_{BC} + \Theta(-l)\partial_D\Gamma^{A-}_{BC} \\ &= \Theta(l)\partial_D\Gamma^{A+}_{BC} + \Theta(-l)\partial_D\Gamma^{A-}_{BC} + \delta(l)n_D\left[\Gamma^A_{BC}\right]\end{aligned} \tag{4.13}$$

Using (4.13), the Riemann tensor is found as

$$R^A_{BCD} = \Theta(l)R^{A+}_{BCD} + \Theta(-l)R^{A-}_{BCD} + \delta(l)\left(\left[\Gamma^A_{BD}\right]n_C - \left[\Gamma^A_{BC}\right]n_D\right) \tag{4.14}$$

The last term in (4.14) implies a discontinuous curvature across the hypersurface. We will investigate further, and if possible, eliminate it. We first try to express the jump of the connection. If the metric in the overlapping coordinates is continuous across Σ_z, its tangential derivatives are also continuous. So, the jump of the derivative of the metric should be normal to the hypersurface

$$\left[g_{AB,C}\right] = \kappa_{AB}n_C \tag{4.15}$$

where κ_{AB} is the jump of the metric's derivative in the normal direction. Now, the jump of the connection expressed in terms of κ_{AB} is

$$\left[\Gamma^A_{BC}\right] = \frac{1}{2}\left(\kappa^A_B n_C + \kappa^A_C n_B - \kappa_{BC}n^A\right) \tag{4.16}$$

Using this in (4.14) yields

$$R^{A}_{BCD} = \Theta(l) R^{A+}_{BCD} + \Theta(-l) R^{A-}_{BCD} + \frac{1}{2}\delta(l)\left(\kappa^{A}_{D} n_B n_C - \kappa^{A}_{C} n_B n_D - \kappa_{BD} n^A n_C + \kappa_{BC} n^A n_D\right) \tag{4.17}$$

Contracting indices A and C in (4.17) gives the Ricci tensor

$$R_{AB} = \Theta(l) R_{AB}{}^{+} + \Theta(-l) R_{AB}{}^{-} + \delta(l)\left(\kappa_{MA} n^M n_B + \kappa_{MB} n^N n_A - \kappa n_A n_B - \kappa_{AB}\right) \tag{4.18}$$

where $\kappa \equiv \kappa^{A}_{A}$.

Contracting (4.18) with the overlapping region metric (4.9) yields the scalar curvature

$$R = \Theta(l) R^{+} + \Theta(-l) R^{-} + \delta(l)\left(\kappa_{MN} n^M - \kappa\right) \tag{4.19}$$

Finally, we find the Einstein tensor as

$$G_{AB} = \Theta(l) G_{AB}{}^{+} + \Theta(-l) G_{AB}{}^{-} + \frac{1}{2}\delta(l)\left(\kappa_{MA} n^M n_B + \kappa_{MB} n^M n_A - \kappa n_A n_B - \kappa_{AB} - g_{AB}\kappa_{MN} n^M n^N + g_{AB}\kappa\right) \tag{4.20}$$

Using the five dimensional Einstein field equations we obtain an expression for the stress-energy tensor

$$T_{AB} = \Theta(l) T_{AB}{}^{+} + \Theta(-l) T_{AB}{}^{-} + \delta(l)\tau_{AB} \tag{4.21}$$

where τ_{AB} is defined as

$$\kappa_5{}^2 \tau_{AB} \equiv \frac{1}{2}\left(\kappa_{MA} n^M n_B + \kappa_{MB} n^M n_A - \kappa n_A n_B - \kappa_{AB} - g_{AB}\kappa_{MN} n^M n^N + g_{AB}\kappa\right) \tag{4.22}$$

where $\kappa_5{}^2$ is the constant of proportionality in five dimensional field equations. Note that the last term in (4.21) is the stress-energy tensor resulting from a thin layer, in other words, the stress-energy tensor of the hypersurface is τ_{AB}. Noting that $\tau_{AB}n^B = 0$, we conclude that τ_{AB} is tangent to the hypersurface

$$\tau_{AB} = \tau_{ij} e^i_A e^j_B \tag{4.23}$$

Using this in (4.22) yields

$$\begin{aligned}\kappa_5^2 \tau_{ij} &= -\frac{1}{2}\kappa_{AB} e^A_i e^B_j - \frac{1}{2} h_{ij} \kappa_{MN} n^M n^N + \frac{1}{2} h_{ij} \kappa_{MN} g^{MN} \\ &= -\frac{1}{2}\kappa_{AB} e^A_i e^B_j + \frac{1}{2} h_{ij} h^{kl} \kappa_{MN} e^M_k e^N_l\end{aligned} \tag{4.24}$$

where we have used (3.13).

On the other hand, we have

$$\begin{aligned}\left[\nabla_B n_A\right] &= -\left[\Gamma^C_{AB}\right] n_C \\ &= \frac{1}{2}\left(\kappa_{AB} - n_C\left(\kappa^C_A n_B + \kappa^C_B n_A\right)\right)\end{aligned} \tag{4.25}$$

Projecting (4.25) on the hypersurface, we get the jump of the extrinsic curvature across Σ_z

$$\left[K_{ij}\right] = \left[\nabla_B n_A\right] e^B_i e^A_j = \frac{1}{2}\kappa_{AB} e^B_i e^A_j \tag{4.26}$$

Comparing (4.26) with (4.24), we express the stress-energy tensor of the hypersurface in terms of the jump of the extrinsic curvature

$$\tau_{ij} = -\frac{1}{\kappa_5^2}\left(\left[K_{ij}\right] - h_{ij}\left[K\right]\right) \tag{4.27}$$

To have a continuos curvature across the boundary then means to have a non-jumping extrinsic curvature on the hypersurface. Setting

$$[K_{ij}] = 0 \tag{4.28}$$

makes sure that there is no discontinuity at the hypersurface. The equation (4.28) is called *the second junction condition.*

4.4 Junction Conditions in Brane World Scenarios

For an infinitesimally thin brane, the first condition (4.11) can be applied, as the bulk should be a valid solution of the field equations. But in the context of brane world scenarios, one has a non-zero brane tension and localized matter fields, and the second condition (4.28) is not applicable. Nevertheless, the result of (4.27) will serve us well in the task of expressing the effective four dimensional field equations. As we will see, it is possible to uniquely determine the extrinsic curvature in terms of the energy momentum tensor on the hypersurface, by imposing $\mathbb{Z}_2$ symmetry [42]. But first, let's reverse the equation (4.27) and determine the jump in the extrinsic curvature in terms of the stress-energy tensor by tracing it, then placing [K] in (4.27) again. We find

$$\left[K_{ij}\right] = -\kappa_5^2\left(\tau_{ij} - \frac{1}{3}\tau h_{ij}\right) \tag{4.29}$$

If we impose $\mathbb{Z}_2$ symmetry with the hypersurface as the fixed point, the normal vector will change direction from the hypersurface point of view, because it always points towards M_5^+. In M_5^- , it points into the hypersurface, whereas in M_5^+ it points out of the hypersurface. Using the definition of the extrinsic curvature, we deduce

$$K_{ij}{}^{+} = -K_{ij}{}^{-} = -\frac{1}{2\kappa_5^2}\left(\tau_{ij} - \frac{1}{3}h_{ij}\tau\right) \tag{4.30}$$

In the next chapter, we will express (3.45) in terms of the stress-energy tensor, and determine what shape the field equations take.

5. GRAVITATIONAL FIELD EQUATIONS ON A 3-BRANE

5.1 Effective Field Equations

Now having every tool that we need, we will obtain the more general (than [42]) field equations. Note that the choice (2.1) is identical to setting

$$N^2 - 1 = N^i = 0 \tag{5.1}$$

Using the equations we calculated in Chapter 3 for dimensional reduction from five to four dimensions, we find the effective field equations on a 3-brane. To do this, we first express the intrinsic Einstein tensor in terms of the bulk stress-energy tensor. Inspecting (3.45)**,** one can see that it contains derivatives with respect to the normal coordinate which, along with the Lie derivative with respect to the shift vector, can be written as a normal vector Lie Derivative. Those terms contain the bulk's effects on gravity on the brane and we express them in a simpler and cleaner form.

Among equations (3.35)-(3.37), only (3.37) involves terms describing the evolution into the fifth dimension. We have to replace it with four dimensional quantities on the brane. We note that

$$R_{5j5l} = R_{ABCD} Z^A Z^C e_j^B e_l^D \tag{5.2}$$

which, by means of (3.11) can be written as

$$R_{5j5l} = N^2 \tilde{E}_{jl} + N^k \left(R_{5jkl} + R_{5lkj} \right) - N^i N^k R_{ijkl} \tag{5.3}$$

where $\tilde{E}_{jl} = R_{ABCD} n^A n^C e_j^B e_l^D$.

One can also use (3.36) to present (5.3) in the form

$$R_{5j5l} = N^2 \tilde{E}_{jl} + R_{5jkl} N^k - N N^k \left(D_k K_{jl} - D_j K_{kl} \right) \tag{5.4}$$

Comparing (5.4) with (3.37), we obtain

$$-\frac{1}{N}\left(\partial_5 K_{jl} - \mathfrak{L}_{\vec{N}} K_{jl} \right) = \tilde{E}_{jl} - K_j^m K_{ml} + \frac{1}{N} D_l D_j N \tag{5.5}$$

We also note that

$$\begin{aligned} R_{5jkl} &= R_{ABCD} Z^A e_j^B e_k^C e_l^D \\ &= R_{ijkl} N^i + N \tilde{B}_{jkl} \end{aligned} \tag{5.6}$$

where $\tilde{B}_{jkl} = R_{ABCD} n^A e_j^B e_k^C e_l^D$.

Comparing (5.6) with (3.36) we see that

$$\tilde{B}_{jkl} = 2 D_{[l} K_{k]j} \tag{5.7}$$

It is also worth to note that

$$G_{i5} = G_{iA} Z^A = G_{ik} N^k + N G_{iA} n^A \tag{5.8}$$

comparing (5.8) with (3.46) we see that

$$G_{iA} n^A = D_m K_i^m - D_i K \tag{5.9}$$

and finally note that

$$G_{55} = N^2 G_{AB} n^A n^B + 2 N G_{iA} N^i n^A + G_{ik} N^k N^i \tag{5.10}$$

From equations (5.10) and (3.47) we conclude that

$$G_{AB} n^A n^B = -\frac{1}{2}\left({}^{(4)}R - K^2 + K_{im} K^{im} \right) \tag{5.11}$$

From equation (3.43) for scalar curvature, we have

$$\frac{1}{N}\left(\partial_5 - \mathfrak{L}_{\vec{N}}\right)K = \frac{1}{2}\left({}^{(4)}R - R - K_{im}K^{im} - K^2\right) - \frac{1}{N}D_m D^m N \tag{5.12}$$

Taking into account that

$$\frac{1}{N}\left(\partial_5 - \mathfrak{L}_{\vec{N}}\right)Kh_{ik} = \frac{1}{N}h_{ik}\left(\partial_5 K - \mathfrak{L}_{\vec{N}}K\right) + 2KK_{ik} \tag{5.13}$$

and combining (5.5) with (5.12), we obtain

$$\begin{aligned}-\frac{1}{N}\left(\partial_5 - \mathfrak{L}_{\vec{N}}\right)\left(K_{ik} - Kh_{ik}\right) = \tilde{E}_{ik} + \frac{1}{2}h_{ik}\left({}^{(4)}R - R - K_{ms}K^{ms} - K^2\right) - K_i^m K_{mk} \\ +2KK_{ik} + \frac{1}{N}\left(D_i D_k N - h_{ik}D_m D^m N\right)\end{aligned} \tag{5.14}$$

Next, substituting (5.14) into expression (3.45), we transform it into the form

$${}^{(4)}G_{ik} = G_{ik} - \left(\tilde{E}_{ik} - \frac{1}{2}h_{ik}R\right) - \frac{1}{2}h_{ik}{}^{(4)}R - K_i^m K_{mk} + KK_{ik} \tag{5.15}$$

This equation is dimension independent.

Using the decomposition of the Riemann tensor in d dimensions

$$R_{ABCD} = C_{ABCD} + \frac{2}{d-2}\left(g_{A[C}R_{D]B} - g_{B[C}R_{D]A}\right) - \frac{2}{(n-1)(n-2)}R\, g_{A[C}g_{D]B} \tag{5.16}$$

for $d = 5$, we pass from $\tilde{E}_{ij}$ to E_{ij} . As a result, we have

$$\tilde{E}_{ij} = E_{ij} + \frac{1}{3}\left[R_{ij} + \frac{1}{4}h_{ij}R - \frac{1}{2}h_{ij}\left({}^{(4)}R - K^2 + K_{mn}K^{mn}\right)\right] \tag{5.17}$$

where $E_{ij} = C_{ABCD}n^A n^C e_i^B e_j^D$.

Let us write down the 5D Einstein field equations

$$G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R = \kappa_5^2 T_{AB} - \Lambda_5 g_{AB} \tag{5.18}$$

where Λ_5 is the bulk cosmological constant and

$$T_{AB} = {}^{(5)}T_{AB} + \tau_{AB} \delta(z) \tag{5.19}$$

where ${}^{(5)}T_{AB}$ is any energy-momentum tensor of the bulk. Thus, in general, instead of (5.18), we have

$$G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2 \left({}^{(5)}T_{AB} + \sqrt{\frac{h}{g}} \; \tau_{AB} \delta(z) \right) \tag{5.20}$$

where in some cases, τ_{AB} can be presented as

$$\begin{aligned} \tau_{AB} &= -\lambda h_{AB} + S_{AB} \\ \tau_{AB} n^A &= 0 \\ \tau_{ij} &= e_i^A e_j^B \tau_{AB} = -\lambda h_{ij} + S_{ij} \end{aligned} \tag{5.21}$$

Next, we return to the equations (3.45)-(3.47). We have

$$\begin{aligned} G_{ik} &- \frac{1}{N} \left[\left(\partial_5 - \mathfrak{L}_{\vec{N}} \right) \left(K_{ik} - h_{ik} K \right) + D_i D_k N \right] + 2 K_i^m K_{mk} - 3 K K_{ik} \\ &+ \frac{1}{2} h_{ik} \left(K^2 + K_{ms} K^{ms} + \frac{2}{N} D_m D^m N \right) \\ &= -\Lambda_5 h_{ik} + \kappa_5^2 \left({}^{(5)}T_{ik} + \sqrt{\frac{h}{g}} \tau_{ik} \delta(z) \right) \end{aligned} \tag{5.22}$$

$$D_m K_i^m - D_i K = \kappa_5^2 \; {}^{(5)}T_{iB} n^B \tag{5.23}$$

$$\frac{1}{2} \left({}^{(4)}R - K^2 + K_{mn} K^{mn} \right) = \Lambda_5 - \kappa_5^2 \; {}^{(5)}T_{AB} n^A n^B \tag{5.24}$$

Now, going *on* the brane and substituting (5.14) into (5.22) and (5.24), we have

$$\begin{aligned} &{}^{(4)}G_{ik} + \frac{1}{2}h_{ik}\left(K^2 - K_{ms}K^{ms}\right) + K_i^m K_{mk} - KK_{ik} + \tilde{E}_{ik} \\ &= \frac{1}{3}h_{ik}\Lambda_5 + \kappa_5^2\left[{}^{(5)}T_{ik} + h_{ik}\left({}^{(5)}T_{AB}n^A n^B - \frac{1}{3}\,{}^{(5)}T\right)\right] \end{aligned} \tag{5.25}$$

It may be useful to introduce the traceless tensor W_{ik} as

$$W_{ik} = \tilde{E}_{ik} - \frac{1}{4}h_{ik}\tilde{E} \tag{5.26}$$

where

$$\tilde{E} = h^{ik}\tilde{E}_{ik} = \frac{1}{2}\left(R - {}^{(4)}R - K_{ms}K^{ms} + K^2\right) \tag{5.27}$$

Then (5.25) can be written in the following form

$$\begin{aligned} &G_{ik} + \left(K_i^m K_{mk} - \frac{1}{2}h_{ik}K_{ms}K^{ms}\right) - K\left(K_{ik} - \frac{1}{2}h_{ik}K\right) + W_{ik} \\ &= -\frac{1}{2}\Lambda_5 h_{ik} + \kappa_5^2\left[{}^{(5)}T_{ik} - \frac{1}{4}h_{ik}\left({}^{(5)}T - 3\,{}^{(5)}T_{AB}n^A n^B\right)\right] \end{aligned} \tag{5.28}$$

where

$$\begin{aligned} W_{ik} = &-\frac{1}{N}\left[\left(\partial_5 - \mathfrak{L}_{\vec{N}}\right)\left(K_{ik} - \frac{1}{4}Kh_{ik}\right) + D_i D_k N\right] - \frac{1}{2}KK_{ik} + K_i^m K_{mk} \\ &+ \frac{1}{4}h_{ik}\left(K_{ms}K^{ms} + \frac{1}{N}D_m D^m N\right) \end{aligned} \tag{5.29}$$

To relate the tensor W_{ik} to E_{ik}, one can use (5.17) which can be put in the form

$$E_{ik} = W_{ik} + U_{ik} \tag{5.30}$$

where

$$
\begin{aligned}
U_{ik} &= -\frac{1}{3}R_{ik} + \frac{1}{24}h_{ik}\left(R + {}^{(4)}R - K^2 + K_{ms}K^{ms}\right) \\
&= -\frac{\kappa_5^2}{3}\left({}^{(5)}T_{ik} - \frac{1}{4}h_{ik}h^{ms}\,{}^{(5)}T_{ms}\right)
\end{aligned} \tag{5.31}
$$

We see that $U_{ik} = 0$ if the bulk energy-momentum tensor vanishes and W_{ik} coincides with E_{ik} .

Next, we shall discuss the conservation equations. We begin with

$$
D_m K_i^m - D_i K = \kappa_5^2\ {}^{(5)}T_{iB}n^B = \kappa_5^2 J_i \tag{5.32}
$$

Substituting into this equation (4.30), we obtain

$$
D_m \tau_i^m = -2J_i \tag{5.33}
$$

It follows that there is exchange of energy flux between the brane and the bulk. When $J_i = 0$, we arrive at the conservation equation for matter on brane. On the other hand, calculating the divergence of (5.28), we obtain the equation

$$
D^i W_{ik} = K^{mi}\tilde{B}_{kim} - \kappa_5^2\left(K_k^i J_i - KJ_k\right) + \frac{1}{2}\kappa_5^2 h_{ik} D^i \rho - 2D^i U_{ik} \tag{5.34}
$$

where $\rho = {}^{(5)}T_{AB}n^A n^B$, or substituting into this equation, the relation (4.30), we obtain its alternative form

$$
\begin{aligned}
D^i W_{ik} &= \frac{1}{4}\kappa_5^4\left[\tau^{mi}\left(D_k\tau_{mi} - D_i\tau_{mk}\right) + \frac{1}{3}\left(\tau_k^i - \delta_k^i\tau\right)D_i\tau + 2J_i\left(\tau_k^i - \frac{1}{3}\delta_k^i\tau\right)\right] \\
&+ \frac{1}{2}\kappa_5^2 D_k\rho - 3D^i U_{ik}
\end{aligned} \tag{5.35}
$$

We see that when the bulk energy momentum tensor vanishes, the divergence of W_{ik} is completely determined by the matter distribution on the brane. We have

$$ {}^{(4)}G_{ik} = -\frac{1}{2}h_{ik}\left(\Lambda_5 - \kappa_5^2\rho\right) - W_{ik} - 3U_{ik} + \pi_{ik}\kappa_5^4 \tag{5.36} $$

where

$$ \pi_{ik} = -\frac{1}{4}\left[\tau_i^m\tau_{mk} - \frac{1}{3}\tau\tau_{ik} - \frac{1}{2}h_{ik}\left(\tau_{ms}\tau^{ms} - \frac{1}{3}\tau^2\right)\right] \tag{5.37} $$

Supposing that

$$ \tau_{ik} = -\lambda h_{ik} + S_{ik} \tag{5.38} $$

where λ is the brane tension, we have, instead of (5.36), the following equation

$$ {}^{(4)}G_{ik} = -\Lambda h_{ik} + \kappa_4^2 S_{ik} + \kappa_5^4\tilde{\pi}_{ik} - W_{ik} - 3U_{ik} \tag{5.39} $$

where,

$$ \Lambda_4 = \frac{1}{2}\left(\Lambda_5 + \frac{1}{6}\kappa_5^4\lambda^2 - \kappa_5^2\rho\right) \tag{5.40} $$

$$ \kappa_4^2 = \frac{1}{6}\lambda\kappa_5^2 \tag{5.41} $$

$$ \tilde{\pi}_{ik} = -\frac{1}{4}\left[\left(S_i^m S_{mk} - \frac{1}{3}SS_{ik}\right) - \frac{1}{2}h_{ik}\left(S_{ml}S^{ml} - \frac{1}{3}S^2\right)\right] \tag{5.42} $$

In the absence of the bulk energy momentum, equations (5.39)-(5.42) coincide with the result of [42].

On the other hand, if the pressure on the brane ρ is constant that Λ_4 can be thought of as 4D cosmological constant with contribution from bulk energy-matter. However, if the brane matter tensor $S_{ik} = 0$, and $\Lambda_4 = 0$, we have equations

$$ {}^{(4)}R_{ik} = -\left(W_{ik} + 3U_{ik}\right) \tag{5.43} $$

and

$$ D^i\left(W_{ik} + 3U_{ik}\right) = 0 \tag{5.44} $$

If we denote the 4D general relativity energy-momentum tensor $\kappa_4^2 \; {}^{(4)}T_{ik} \leftrightarrow -\left(W_{ik} + 3U_{ik}\right)$ we see that a stationary general relativity solution with traceless energy momentum tensor gives rise to non-vacuum brane-world solution in five dimensional gravity with traceless brane-on components of the bulk energy momentum tensor.

5.2 Evolution Equations

The tensor W_{ij} of (5.36) is not freely specifiable, but its divergence is constrained by matter terms. This means that our effective field equations are not closed. To overcome this we will find the evolution equations describing the evolution of W_{ik} in the bulk.

The five dimensional Bianchi identities $\nabla_{[A}R_{BC]DE} = 0$ reduce to four independent equations

$$ \nabla_{[i}R_{jk]lm} = 0 \tag{5.45} $$

$$ \nabla_{[5}R_{jk]lm} = 0 \tag{5.46} $$

$$ \nabla_{[i}R_{jk]l5} = 0 \tag{5.47} $$

$$ \nabla_{[i}R_{j5]l5} = 0 \tag{5.48} $$

These equations give

$$D_{[i}{}^{(4)}R_{jk]lm}=0 \tag{5.49}$$

$$\begin{aligned}\left(\partial_5-\mathfrak{L}_{\vec{N}}\right){}^{(4)}R_{ijkl}+2N\,{}^{(4)}R_{ijm[k}K_{l]}^{m}+2ND_{[i}\tilde{B}_{|kl|j]}+2\tilde{B}_{ij[l}D_{k]}N \\ +2\tilde{B}_{kl[j}D_{i]}N+2K_{j[l}D_{k]}D_iN+2K_{i[k}D_{l]}D_jN=0\end{aligned} \tag{5.50}$$

$$D_{[i}\tilde{B}_{kj]l}+K_{[i}^{m}\,{}^{(4)}R_{kj]ml}=0 \tag{5.51}$$

$$\begin{aligned}\left(\partial_5-\mathfrak{L}_{\vec{N}}\right)\tilde{B}_{jik}+2ND_{[j}\tilde{E}_{i]k}-NK_k^m\tilde{B}_{jim}+2N\tilde{B}_{km[j}K_{i]}^{m} \\ +2\tilde{E}_{k[i}D_{j]}N+{}^{(4)}R_{ijmk}D^mN=0\end{aligned} \tag{5.52}$$

Using the 5D field equations (5.18) in (5.52) we get

$$\begin{aligned}&\left(\partial_5-\mathfrak{L}_{\vec{N}}\right)B_{jik}+2ND_{[j}W_{i]k}-NK_k^mB_{jim}+{}^{(5)}C_{imjk}D^mN+2NB_{km[j}K_{i]}^{m} \\ &+2W_{k[i}D_{j]}N+\frac{2}{3}N\kappa_5^2J_mK_{[i}^{m}h_{j]k}+\frac{2}{3}N\kappa_5^2J_{[i}K_{j]k}+\frac{2}{3}\kappa_5^2h_{k[j}\left(\partial_5-\mathfrak{L}_{\vec{N}}\right)J_{i]} \\ &+\frac{N}{2}\kappa_5^2h_{k[i}\left(D_{j]}\rho-\frac{1}{3}D_{j]}T\right)+2U_{k[i}D_{j]}N+2h_{k[i}U_{j]m}D^mN+\frac{1}{6}\kappa_5^2Th_{k[j}D_{i]}N \\ &+\frac{5}{6}\kappa_5^2\rho h_{k[i}D_{j]}N=0\end{aligned} \tag{5.53}$$

Again, using (5.18) in (5.50), we get

$$\begin{aligned}\left(\partial_5-\mathfrak{L}_{\vec{N}}\right)W_{ij}=&-3\left(\partial_5-\mathfrak{L}_{\vec{N}}\right)U_{ij}-\frac{1}{2}\kappa_5^2h_{ij}\left(\partial_5-\mathfrak{L}_{\vec{N}}\right)\left(\rho+\frac{1}{3}T\right)+NK^{mk}\,{}^{(4)}R_{mikj} \\ &+3NK_j^mU_{im}+2NK_{(i}^{m}W_{j)m}+NK_j^mW_{im}+NK^{km}\left(K_{ik}K_{jm}-K_{ij}K_{mk}\right) \\ &+2ND_{[k}\tilde{B}^{k}{}_{|j|i]}+2\tilde{B}^{k}{}_{i[j}D_{k]}N+2\tilde{B}^{k}{}_{j[i}D_{k]}N-K_{(i}^{k}D_{j)}D_kN-NKW_{ij} \\ &-\frac{N}{4}\rho\kappa_5^2\left(h_{ij}K+K_{ij}\right)+\frac{N}{6}T\kappa_5^2\left(\frac{1}{2}Kh_{ij}+K_{ij}\right)+\frac{N}{6}\Lambda_5\left(K_{ij}-Kh_{ij}\right)\end{aligned} \tag{5.54}$$

Those two evolution equations, along with (5.36) form a closed system of equations.

6. GRAVITATIONAL FIELD EQUATIONS ON A 2-BRANE

6.1 Effective Field Equations

The equations calculated in Chapter 3 are actually independent of the number of dimensions. Using those, we repeated the method of Chapter 5 for a 2-brane embedded in a four dimensional manifold, and found the effective three dimensional field equations. Note that only in this chapter, all quantities are assumed four dimensional except where indicated.

The gravitational field equations on a 2-brane are

$$ {}^{(3)}G_{ik} = -\Lambda_3 h_{ik} + \kappa_3^2 S_{ik} + \kappa_4^4 \tilde{\pi}_{ik} - W_{ik} - 2U_{ik} \tag{6.1} $$

where

$$ \Lambda_3 = \frac{1}{3}\left(\Lambda_4 - \kappa_4^2 \rho + \frac{3}{16}\kappa_4^2 \lambda^2 \right) \tag{6.2} $$

$$ \kappa_3^2 = \frac{1}{8}\kappa_4^4 \lambda \tag{6.3} $$

$$ \tilde{\pi}_{ik} = -\frac{1}{4}\left[S_i^m S_{mk} - \frac{1}{2} S S_{ik} - \frac{1}{2} h_{ik} \left(S_{mn} S^{mn} - \frac{1}{2} S^2 \right) \right] \tag{6.4} $$

$$ U_{ik} = -\frac{1}{2}\kappa_4^2 \left({}^{(4)}T_{ik} - \frac{1}{3} h_{ik} h_{mn} {}^{(4)}T^{mn} \right) \tag{6.5} $$

$$
\begin{aligned}
W_{ik} &= \tilde{E}_{ik} - \frac{1}{3} h_{ik} \tilde{E} \\
&= -\frac{1}{N}\left[\left(\partial_4 - \mathfrak{L}_{\vec{N}}\right)\left(K_{ik} - \frac{1}{3} h_{ik} K\right) + D_i D_k N\right] - \frac{2}{3} K K_{ik} + K_i^m K_{mk} \\
&+ \frac{1}{3} h_{ik}\left(K_{mn} K^{mn} + \frac{1}{N} D_m D^m N\right)
\end{aligned} \tag{6.6}
$$

6.2 Evolution Equations

To make the three dimensional field equations closed, we repeat the calculations of Section 5.2 for a 2-brane thus finding

$$
\begin{aligned}
&\left(\partial_4 - \mathfrak{L}_{\vec{N}}\right) B_{jik} + 2N D_{[j} W_{i]k} - N K_k^m B_{jim} + 2N B_{km[j} K_{i]}^m + 2N B_{km[j} K_{i]}^m \\
&\quad + 2 W_{k[i} D_{j]} N + N \kappa_4^2 J_m K_{[i}^m h_{j]k} + N \kappa_4^2 J_{[i} K_{j]k} + \kappa_4^2 h_{k[j}\left(\partial_4 - \mathfrak{L}_{\vec{N}}\right) J_{i]} \\
&\quad + \frac{2N}{3} \kappa_4^2 h_{k[i}\left(D_{j]} \rho - \frac{1}{2} D_{j]} T\right) + C_{ijmk} D^m N + 2 U_{k[i} D_{j]} N \\
&\quad + 2 h_{k[i} U_{j]m} D^m N + \frac{1}{3}(4\rho - T) \kappa_4^2 h_{k[i} D_{j]} N
\end{aligned} \tag{6.7}
$$

and

$$
\begin{aligned}
\left(\partial_4 - \mathfrak{L}_{\vec{N}}\right) W_{ij} &= 2N D_{[k} \tilde{B}^k{}_{|i|j]} - 2\left(\partial_4 - \mathfrak{L}_{\vec{N}}\right) U_{ij} + N K^{mn} R_{imjn} + 2N K_i^m U_{jm} \\
&+ 2N K_i^m U_{jm} + \tilde{B}^k{}_{j[i} D_{k]} N + 2 \tilde{B}^k{}_{i[j} D_{k]} N + 2N W_{m(i} K_{j)}^m - N K K_i^m K_{mj} \\
&- N K^{mn} K_{mn} K_{ij} - N K W_{ij} - \frac{2N}{3} \kappa_4^2 \left(\partial_4 - \mathfrak{L}_{\vec{N}}\right) \rho \\
&- \frac{4N}{3} \kappa_4^2 \rho \left(K_{ij} + \frac{1}{4} K h_{ij}\right) + \frac{N}{3} \kappa_4^2 T \left(K_{ij} + \frac{1}{2} K h_{ij}\right) \\
&- \frac{1}{3} N K \Lambda_4 h_{ij}
\end{aligned} \tag{6.8}
$$

7. ROTATING BLACK HOLE ON A RS2 BRANE

We will, using the approach of [54], solve the constraint equations to get a rotating black hole solution on the brane.

7.1 Black Hole Without Matter on Brane

The Hamiltonian constraint equation of (5.11) is

$$\kappa_5^2 T_{AB} = -\frac{1}{2}\left({}^{(4)}R - K^2 + K_{mn}K^{mn}\right) \tag{7.1}$$

For a Randall Sundrum brane we have

$$\begin{aligned} \Lambda_5 &= -\frac{6}{l^2} \\ \kappa_5^2 &= 2l\kappa_4^2 \\ \lambda &= \frac{6}{l\kappa_5^2} \end{aligned} \tag{7.2}$$

If the energy momentum tensor of the bulk and the brane vanishes everywhere, equation (7.1) reduces to

$${}^{(4)}R = 0 \tag{7.3}$$

To solve (7.3) for a rotating black hole, we propose a metric ansatz in Kerr Schild form [56] as

$$\begin{aligned} ds^2 = &-\left(du + dr\right)^2 + dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta d\tilde{\phi}^2 + 2a\sin^2\theta drd\tilde{\phi} \\ &+h\left(r,\theta\right)k_i k_j dx^i dx^j \end{aligned} \tag{7.4}$$

where a is the rotation parameter, $k_i = (1, 0, 0, - a \sin^2\theta)$ is a null vector and

$$\Sigma = r^2 + a^2 \cos^2 \theta \tag{7.5}$$

$$du = dt - \frac{\left(r^2 a^2\right)}{r^2 + a^2 - 2Mr} dr \tag{7.6}$$

$$d\tilde{\phi} = d\phi - \frac{a}{r^2 + a^2 - 2Mr} dr \tag{7.7}$$

Using (7.4) and solving (7.3) for $h\left(r,\theta\right)$ gives

$$h\left(r,\theta\right) = -\frac{\beta}{\Sigma} + \frac{2Mr}{\Sigma} \tag{7.8}$$

where, β and M are integration constants. (7.8) is a Kerr type solution with a charge term. This *tidal charge* is due to non-local bulk effects resulting from tensor W_{ij}.

7.2 Black Hole with Gauge Field on Brane

Trapping a Maxwell field on the brane with potential one form

$$A = -\frac{er}{\Sigma}\left(du - a \sin^2 \theta d\tilde{\phi}\right) \tag{7.9}$$

will give a four dimensional energy-momentum tensor as

$$\kappa_4^2 T_{ij} = F_{ik} F_j^k - \frac{1}{4} h_{ij} F_{km} F^{km} \tag{7.10}$$

In this case equation (7.1) reduces to

$${}^{(4)}R = -\frac{le^4}{\Sigma^4} \tag{7.11}$$

Using the ansatz (7.4), we find a general solution to (7.11) as

$$h(r,\theta) = -\frac{e^2+\beta}{\Sigma} + \frac{2Mr}{\Sigma} - le^4\left(\frac{a\cos\theta\left(2a^4\cos^4\theta + 5a^2r^2\cos^2\theta + 3r^4\right) + 3r\Sigma^2\arctan\left(r/a\cos\theta\right)}{8\Sigma^3 a^5\cos^5\theta}\right) \tag{7.12}$$

which, up to linear terms in $a^2\cos^2\theta$ gives

$$h(r,\theta) \approx -\frac{e^2+\beta}{r} + \frac{2Mr}{r^2} - \frac{le^4}{20r^6} + \frac{a^2\cos^2\theta}{r^2}\left(\frac{e^2+\beta}{r^2} - \frac{2M}{r} + \frac{17le^4}{140r^6}\right) \tag{7.13}$$

Setting $a = 0$ in (7.13), one recovers the non-rotating solution of [54].

8. CONCLUSION

We have derived covariant gravitational field equations on 2-brane and 3-brane worlds in the framework of non-geodesic slicing of bulk spacetime.

We have shown that the form of these effective equations on the branes remains the same as in the case of geodesic slicing of bulk spacetime [42] and coincides exactly with them when acceleration vanishes.

Since our effective gravitational equations, as in the case of geodesic slicing, are not closed, we have derived the evolutionary equations into the bulk identites which make our system of equations closed.

Thus the gravitational field equations obtained in this work, generalize the equations of Shiromizu, Maeda and Sasaki [42], removing the special presuggestion of Gaussian normal coordinates.

Further, we have studied black hole solutions on a 3-brane, solving the constraint equations for a rotating black hole. We have found that the original metric of rotating black hole acquires a tidal charge on the 3-brane due to the bulk effect of Weyl curvature tensor.

REFERENCES

[1] **Overduin, J. M. and Wesson, P. S.**, 1997. Kaluza-Klein Gravity, *Phys. Rept*. **283**, 303-380.

[2] **Rubakov**, **V. A. ,** 2001. Large and Infinite Extra Dimensions, *Phys. Usp.* **44**, 871-893.

[3] eds. **Appelquist, T., Chodos, A., Freund, P. G. O.,** 1987. Modern Kaluza Klein Theories, Addison-Wesley, Menlo Park.

[4] **Minkowski, H.,** 1909. Raum und Zeit, *Physik. Zeits*. **10**, 104-117.

[5] **Einstein, A.**, 1916. Die Grundlage der Allgemeinen Relativitätstheorie, *Annalen Der Physik* **49**, 769-822.

[6] **Nordstöm, G.**, 1914. Über die Möglichkeit das Elektromagnetische Feld und das Gravitationsfeld zu vereinigen, *Physik. Zeits.* **15**, 504-506. (English translation in [3]).

[7] **Kaluza, T.**, 1921. Zum Unitätsproblem der Physik, *Sitz. Preuss. Akad. Wiss. Phys. Math.* **K1**, 966-972 (English translation in [3]).

[8] **Klein, O.**, 1926. Quantentheorie un Fünfdimensionale Relativitätstheorie, *Zeits. Phys.* **37,** 895-906 (English translation in [3]).

[9] **Klein, O.**, 1926. The Atomicity of Electricity as a Quantum Theory Law, *Nature* **118**, 516.

[10] **Lykken, J. D.**, 1996. Weak Scale Superstring, *Phys. Rev.* D **54**, 3693-3697.

[11] **Polchinski, J.**, 1996. TASI Lectures on D-Branes, hep-th/9611050.

[12] **Horava, P., Witten, E.**, 1996. Heterotic and Type I String Dynamics from Eleven Dimensions, *Nucl. Phys.* B **460**, 506-524.

[13] **Lukas, A., Ovrut, B. A., Stelle, K. S., Waldram, D.**, 1999. The Universe as a Domain Wall, *Phys. Rev.* D **59**, 086001.

[14] **Akama, K.**, 1982. An Early Proposal of "Brane World" , *Lect. Notes Phys.* **176**, 267-271.

[15] **Rubakov, V. A., Shaposhnikov, M. E.**, 1983. Do We Live Inside a Domain Wall?, *Phys. Lett.* B **125**, 136-138.

[16] **Joseph, D. W.**, 1962. Coordinate Covariance and the Particle Spectrum, *Phys. Rev.* D **126**, 319-323.

[17] **Visser, M.**,1985. An Exotic Class of Kaluza-Klein Models, hep-th/9910093.

[18] **Arkani-Hamed, N., Dimopoulos, S., Dvali, G.**, 1998. The Hierarchy Problem and New Dimensions at a Milimeter, *Phys. Lett.* B **429**, 263-272.

[19] **Antoniadis, I., Arkani-Hamed, N., Dimopoulos, S., Dvali, G.**, 1998. New Dimensions at a Millimeter to a Fermi and Superstrings at a TeV, *Phys. Lett.* B **436**, 257-263.

[20] **Long, J. C.**, 1999. Experimental Status of Gravitational Strength Forces in the Sub-Millimeter Regime, *Nucl. Phys*. B **539**, 23-24.

[21] **Su, Y., Heckel, B. R., Adelberger E. G., Gundlach, J. H., Harris, M., Smith, G. L., Swanson, H. E.**,1994. New Tests of the Universality of Free Fall, *Phys. Rev.* D **50**, 4614-3636.

[22] **Hoyle, C. D., Schmidt, U., Heckel, B. R., Adelbeyer, E. G., Gundlach, J. H., Kapner, D. J., Swanson, H. E.**, 2001. Sub-Millimeter Tests of the Gravitational Inverse-Square Law: A Search for "Large" Extra Dimensions, *Phys. Rev. Lett.* **86**, 1418-1421.

[23] **Long, J. C., Churnside, A. B., Price, J. C.**, 2002. Gravitational Experiments Below 1 Millimeter and Comment on Shielded Casimir Backgrounds for Experiments in the Micron Region, in *Proceedings of the Ninth Marcel Grossman Meeting on General Relativity*, part C, pp. 1825-1826, World Scientific.

[24] **Dimopoulos, S., Geraci, A. A.**, 2003. Probing Sub-Micron Forces by Interferometry of Bose-Einstein Condensed Atoms, *Phys.Rev.* D **68**, 124021.

[25] **Long, J. C., Price, J. C.**, 2003. Current Short-Range Tests of the Gravitational Inverse Square Law, *Comptes Rendus Physique* **4**, 337-346.

[26] **Long, J. C., Chan, H. W., Churnside, A. B., Gulbis, E. A., Varney, M. C. M., Price, J. C.**, 2003. New Experimental Limits on Macroscopic Forces Below 100 Microns, *Nature* **421**, 922-925.

[27] **Fischbach, E., Krause, D. E., Mostepanenko, V. M., Novello, M.**, 2001. New constraints on ultrashort-ranged Yukawa interactions from atomic force microscopy, *Phys.Rev*. D **64**, 075010.

[28] **Randall, L., Sundrum, R.**, 1999. A Large Mass Hierarchy from a Small Extra Dimension, *Phys. Rev. Lett*. **83**, 3370-3373.

[29] **Randall, L., Sundrum, R.**, 1999. An Alternative to Compactification, *Phys. Rev. Lett*. **83**, 4690-4693.

[30] **Lykken, J., Randall, L.**, 2000. The Shape of Gravity, *JHEP* **0006**, 014.

[31] **Karch, A., Randall, L.**, 2001. Locally Localized Gravity, *JHEP* **0105**, 008.

[32] **Perlmutter, S., Aldering, G., Goldhaber, G., Knop, R.A., Nugent, P., Castro, P.G., Deustua, S., Fabbro, S., Goobar, A., Groom, D.E., Hook, I. M., Kim, A.G., Kim, M.Y., Lee, J.C., Nunes, N.J., Pain, R., Pennypacker, C.R., Quimby, R., Lidman, C.,. Ellis, R.S, Irwin, M., McMahon, R.G., Ruiz-Lapuente, P., Walton, N., Schaefer, B., Boyle, B.J., Filippenko, A.V., Matheson, T., Fruchter, A.S., Panagia, N., Newberg, H.J.M., Couch, W.J.**, 1999. Measurements of Omega and Lambda from 42 High-Redshift Supernovae, *Astrophys. J.* **517**, 565-586.

[33] **Riess, A. G., Filippenko, A. V., Challis, P., Clocchiattia, A., Diercks, A., Garnavich, P. M., Gilliland, R. L., Hogan, C. J., Jha, S., Kirshner, R. P., Leibundgut, B., Phillips, M. M., Reiss, D., Schmidt, B. P., Schommer, R. A., Smith, R. C., Spyromilio, J., Stubbs, C., Suntzeff, N. B., Tonry, J.**, 1998. Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, *Astron. J.* **116**, 1009-1038.

[34] **DeWolfe, O., Freedman, D.Z., Gubser, S.S., Karch, A.**, 2000. Modeling the fifth dimension with scalars and gravity, *Phys.Rev.* D **62**, 046008.

[35] **Kaloper, N.**, 1999. Bent Domain Walls as Braneworlds, *Phys.Rev.* D **60**, 123506.

[36] **Kim, H. B., Kim, H. D.**, 2000. Inflation and Gauge Hierarchy in Randall-Sundrum Compactification, *Phys.Rev*. D **61**, 064003.

[37] **Nihei, T.**, 1999. Inflation in the five-dimensional universe with an orbifold extra dimension, *Phys.Lett*. B **465,** 81-85.

[38] **Israel, W.**, 1966. Singular Hypersurfaces and Thin Shells in General Relativity, *Nuovo Cimento* **44B**, 1-14.

[39] **R. M. Wald**, 1984. General Relativity, University of Chicago Press, Chicago.

[40] **Misner, C. W., Thorne, K. S., Wheeler, J. A.**, 1973. Gravitation, W.H. Freeman, New York.

[41] **Poisson, E.**, 2004. A Relativist's Toolkit, Cambridge University Press.

[42] **Shiomizu, T., Maeda, K., Sasaki, M.**, 2000. Einstein Equations on the 3-Brane World, *Phys. Rev.* D **62,** 024012.

[43] **Gregory, R., Laflamme, R.**, 1993. Black Strings and p-branes are Unstable, *Phys. Rev. Lett*. **70**, 2837-2840.

[44] **Chamblin, A., Hawking, S.W., Reall, H.S.**, 2000. Brane-World Black Holes, *Phys.Rev.* D **61**, 065007.

[45] **Emparan, R., Horowitz, G. T., Myers, R. C.**, 2000. Exact Description of Black Holes on Branes, *JHEP* **0001**, 007.

[46] **Emparan, R., Horowitz, G. T., Myers, R. C.**, 2000. Exact Description of Black Holes on Branes II: Comparison with BTZ Black Holes and Black Strings, *JHEP* **0001**, 021.

[47] **Mück, W., Viswanathan, K.S., Volovich, I.V.**, 2000. Geodesics and Newton's Law in Brane Backgrounds, *Phys.Rev*. D **62**, 105019.

[48] **Garriga, J., Tanaka, T.**, 2000. Gravity in the Randall-Sundrum Brane World, *Phys.Rev.Lett*. **84**, 2778-2781.

[49] **Giddings, S.B., Katz, E., Randall, L.**, 2000. Linearized Gravity in Brane Backgrounds, *JHEP* **0003**, 023.

[50] **Duff, M. J., Liu, J. T.**, 2000. Complementarity of the Maldacena and Randall-Sundrum Pictures, *Phys. Rev. Lett*. **85**, 2052-2055.

[51] **Aref'eva, I. Ya., Ivanov, M. G., Mück, W., Viswanathan, K. S., Volovich, I. V.**, 2000. Consistent Linearized Gravity in Brane Backgrounds, *Nucl. Phys*. B **590**, 273-286.

[52] **Giannakis, I., Liu, J. T., Ren, H.**, 2003. Linearized Gravity in the Karch-Randall Braneworld, *Nucl. Phys.* B **654**, 197-224.

[53] **Dadhich, N., Maartens, R., Papadopoulos, P., Rezania, V.**, 2000. Black holes on the Brane, *Phys. Lett.* B **487,** 1-6.

[54] **Chamblin, A., Reall, H. S., Shinkai, H., Shiromizu, T.**, 2001. Charged Brane-World Black Holes, *Phys. Rev.* D **63**, 064015.

[55] **Brecher, D., Perry, M.**, 2000. Ricci-Flat Branes, *Nucl.Phys*. B **566**, 151-172.

[56] **Chandrasekar, S.**, 1992. The Mathematical Theory of Black Holes, Oxford University Press, New York.

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